On the Pedagogy of Randomness: Effectively Teaching How Random Is Relative in High School

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In a 2024 article published in *Nature*, headlined "Probability Probably Doesn't Exist," renowned statistician Sir David Spiegelhalter revisited an idea that is well-known to experienced statisticians but much less so to the general public and, more importantly, to the typical K-12 classroom studying statistics: randomness is relative to the observer. Randomness is not an objective property that some phenomenon inherently possesses but rather is the condition of uncertainty about the phenomenon from the point of view of the people observing it (Spiegelhalter, 2024). Essentially, we deem something random because we, as observers, cannot predict its outcomes with complete accuracy – the less predictable it is to us, the more random it is. Consequently, we assign probabilities to various events based on our limited knowledge of the phenomenon. This idea has been highlighted by numerous statisticians (Gelman, 2012; Kim, 2024; Tan, 2016; UW Video, 2013), and Persi Diaconis gave an excellent lecture on how with enough physical information, a coin toss is actually not random (UW Video, 2013). Yet confusion about the concept persists in statistics education in K-12 and beyond. In this paper, we review this issue and present recommendations with examples on how teachers can present the idea of randomness more clearly across different stages of statistics education.

The Confusion Around Randomness

The confusion around the concept of randomness in statistics education is a known issue. In a study of 82 mathematics student teachers and 45 science student teachers in the United Kingdom, the most common definitions of randomness found were "unpredictable" (38%), "no pattern" (32%), and "no planning/unconscious" (17%) (Ingram, 2022). In the United States, the Common Core State Standards in Mathematics (CCSS-M) use the idea of random in the context of random sampling but do not give an explicit definition of what it means for something to be random (Common Core State Standards Initiative, 2023; Scheaffer & Jacobbe, 2014). On the other hand, the Pre-K-12 Guidelines for Assessment and Instruction in Statistics Education II (GAISE II) more clearly discusses the role of probability in quantifying randomness and provides a comprehensive structure from which to communicate statistical knowledge within this context (Bargagliotti et al., 2020), but likewise does not allocate sufficient space to a discussion of what exactly it mean for something to be random. This issue is exacerbated by the multiple usages of the word random in statistics. For example, we use random sampling to describe a type of correctly selected sample for inferential statistical analysis, such that in contrast, a convenience sample is not a random sample, but the activity of putting together a convenience sample is in fact still a random process, albeit one where we are left with no idea what the underlying sampling distribution is.

One problem is that there is much more focus on what random looks like than what random is. For example, Watson & Fitzallen (2019) discuss how randomness can be explained by using the concepts of variability and unpredictability, which is helpful for recognizing outcomes that would have come from a random process, but is problematic when a student gets into deeper thinking such as, "What if I had an advanced machine that could tell by analyzing all relevant physical variables how a pair of dice would land the moment I throw them? What if this machine can also guide my hand so I can throw the dice exactly as how I would want them to land?" Students can also confound the technical meaning and colloquial meaning of words like randomness that are commonly used outside of Statistics (Kaplan et al., 2014). An example of this can be found in the high school level lesson on random variables in Khan Academy, where some viewers mention being confused between the meanings of "random" and "arbitrary" in the comment section of the video (Khan Academy, 2012). This is again a consequence of focusing on the outcome of random events without considering its relativity to the observer. Asking a person to give an arbitrary number by "thinking of one at random" is different from asking them to give a random number by picking without looking from a jar of numbered balls. The former is not random at all to the person thinking of the number, since the person knows exactly the number they will give before they give it, and they can choose any number they want. Arbitrary simply means that the choice does not have any meaningful purpose, but it is still a fully conscious choice. In contrast, the person has no idea what number they will end up picking from the jar until after they picked it. They will know at the same time as everyone else who looks at the ball with them. Thus, that number is random to them.

Another problem, probably the larger one, is that this issue does not actually cause problems until later, except perhaps to drive students towards a more mechanical approach in learning the subject. That is, it is very possible not to understand that randomness is relative and still know how to compute statistics, how to use statistical tests, and how to follow decision rules for test results. Instead, the problem pops up in misconceptions about how and why statistical tools work and their limitations, and manifests in their misuse and misinterpretation (Strasak et al., 2007).

Addressing this issue around the relativity of randomness and thereby contributing positively to students' better understanding of statistics that could help in making them better scientists is not expected to be laborious. Rather, slight changes in how we introduce topics can shed light on this important idea. Examples of how this can be done are illustrated in the succeeding section.

Proposed Learning Content and Delivery Changes

To illustrate the ideas discussed on changing how the concept of randomness is taught, the following fictional vignettes were constructed. These vignettes were prepared for middle to high school levels, since these are where randomness and other statistical concepts are introduced.

Introducing the Concept of Randomness for the First Time

The concept of randomness should be introduced with the concept of probability, as random events are the phenomenon of interest whereas probability is our way to measure or make sense of it.

Teacher: Today we will talk about the concepts of randomness and probability. Something is random if it is something in the future about which we are not certain. For example, will it rain tomorrow? Who thinks it will rain tomorrow? (Teddy and Joanna raise their hands) Joanna, why do you think it will rain tomorrow?

Joanna: I just have a feeling.

Teacher: How about you, Teddy?

Teddy: My phone says there is a 65% chance of rain tomorrow.

Teacher: Good. Both of you believe that it will rain tomorrow for different reasons, while the rest of the class believe it will not. That belief can be quantified into a number between 0 and 1, where 0 means we are sure the event will not happen, that is it will definitely not rain tomorrow, while 1 means that it will certainly happen. We call this number a probability. The number from Teddy's phone, 65% or 0.65, is a probability. Teddy, do you know how your phone came up with that number?

Teddy: No?

Teacher: That's fine. What we want to do is to learn about reasonable ways to come up with probabilities depending on the situation. Let's do something simple. Here I have a coin. You can see that one side is heads and the other side is tails. Can someone tell me something random we can do with this coin? Ben?

Ben: Toss it?

Teacher: Yes! When we toss the coin, we don't know how it will land. Suppose we are interested in whether it lands heads. What do you think is the probability of that? Lisa?

Lisa: 50-50?

Teacher: Great intuition! Yes, 50-50 or in other words, a probability of 50% or 0.50. You are thinking that because we all know that this coin only has two faces, heads and tails. Thus, heads is one of two possibilities, or 1/2 or 0.50. But, we are making an important assumption here. Can someone tell me what it is? Timothy?

Timothy: That you're not cheating?

Teacher: Exactly! Let me show you two other coins. Here is a two-headed coin and here is a two-tailed coin. For now, I'm going to set these two aside so that we just have this fair coin. So, we are assuming that the coin is fair, and that I'm a fair coin-tosser. If we do that, then what is the probability that a random toss will come up heads? 50%. Alright, let's do a little experiment. I'm going to ask you to write down how you think this coin will come up, then I'll toss it and we'll see if you get it right. Ready? Let's go.

Teacher: Ok, we got tails. Who guessed right? (Joanna, Lisa, and Timothy raise their hand) Congratulations. Lisa, how did you guess it right?

Lisa: I don't know, I just guessed.

Teacher: How about you Joanna? Timothy?

Joanna and Timothy: We also just guessed.

Teacher: Of course. Because under our assumptions, the chances of heads or tails are the same so it doesn't matter what you choose, you have a 50% chance of being right. We each have the same randomness about this experiment. Now let's switch it up a bit. I'm going to take this fair coin and these funny coins I showed you before and put them all in this small box. Now, I need a volunteer. Yes, Timothy come on up here. Ok. I want you to turn around so your classmates can't see your reaction. Now I'm going to pick one of these coins from the box. (Shows two-tailed coin to Timothy). Ok, so I want everyone else to make a guess again and write it down please. Have you written down your guess? Great, here we go.

Teacher: Ok, (without looking at the result) Timothy, what do we have?

Timothy: Tails.

Teacher: That's correct! Thank you, you may take your seat. Ok, who was able to get it right? Not everyone, right? But Timothy definitely got it right. Why is that? Ben?

Ben: Because he knew.

Teacher: Exactly. Timothy and I both knew what the outcome would be even before I tossed the coin, because we knew it was a two-tailed coin. The rest of you did not. What does this tell you about the concept of randomness? Lisa?

Lisa: That it's relative?

Teacher: Yes! How you experience randomness depends on what you know. The rest of you still need to assume that the coin is fair. You have no way of knowing what I picked and you can't read my mind, unless you can… Joanna (a student who guessed right) did you read my mind? No? Great. So in as far as you're concerned, the probability of getting heads is still 0.50. In fact that is still a reasonable assumption if you knew nothing else. Before I actually selected the two-tailed coin, the probability that I would select it is some number between 0 and 1 that you don't know. So without any other information, 0.50 is as good a guess as any. After I decided I was going to pick it, that probability became 1, but only I and Timothy knew this information. So for us, the coin toss outcome was not random at all. But for you, it still fits the definition. It's an event in the future about which YOU are not certain. In our next lessons, this idea of randomness being relative will take a backseat as we talk about how to compute probabilities in different contexts, but it is important to remember that this relative nature of randomness is always there.

Enriching Other Topics
Inclusion of the relative nature of randomness can enrich other topics. Here is an example for expected values, where it is shown how knowing more about how a variable behaves makes it less random to the observer, letting the observer make better decisions.

Teacher: Now that we know how to construct probability distributions and compute expected values, let's consider the following probability distribution table for a six-sided die. What can you tell me about the die represented in Table 1?

Table 1: Probability distribution table for an unfair die

x	1	2	3	4	5	6
P(x)	1/7	1/7	1/7	1/7	1/7	2/7

Teddy: It's rigged?

Teacher: Indeed, it is not a fair die. The number 6 will come up twice as often as any of the other numbers. From our previous activity, we already know that if you select a number from a 6-sided die roll at random, the probability that you will get it right is? Yes, Lisa?

Lisa: 1/6?

Teacher: Correct. So, suppose we do that for dice with this "rigged" distribution and we lose 1 credit each time we fail to guess correctly but gain 4 credits if we guess right, what would be the expected value? Ben?

Ben: 1/6 *4 – 5/6 * 1 = -0.1667

Teacher: Correct. What about if we knew the actual distribution? What strategy should we do in this game? Timothy?

Timothy: Always bet on 6?

Teacher: Exactly. You are all on fire today. If we do it this way, will we always get the right answer? (Students shake their heads no) You're right, we won't. But what will be our expected value? Lisa?

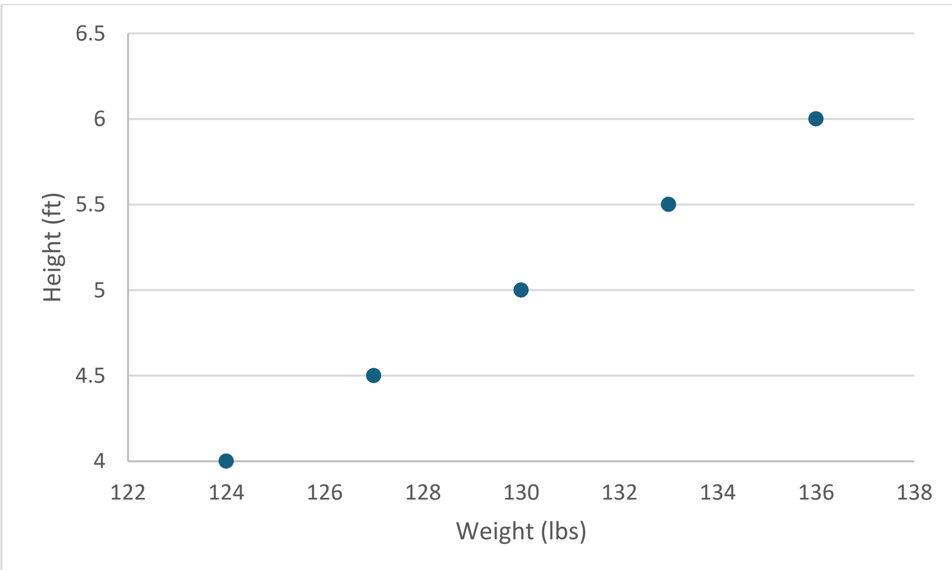
Lisa: 2/7 * 4 – 5/7 * 1 = 0.4286! Which means we will win more than we lose on average and have a good chance of generating more credits as we play more.

Teacher: Exactly, unlike when we didn't know the true distribution where we will lose more as we play more. This is once again showing that randomness is a relative quality that depends on how much information you have. Without any information, we assume the die is fair. With full information, we know it isn't fair. This knowledge does not mean we will get our guess right each time, but it does mean that we have access to the best option. Of course, in many practical cases in real life where we use probability and statistics, we never know the exact true distribution. We can only make the best assumptions we can based on available information.

Introducing Advanced Concepts with Stronger Theoretical Foundation

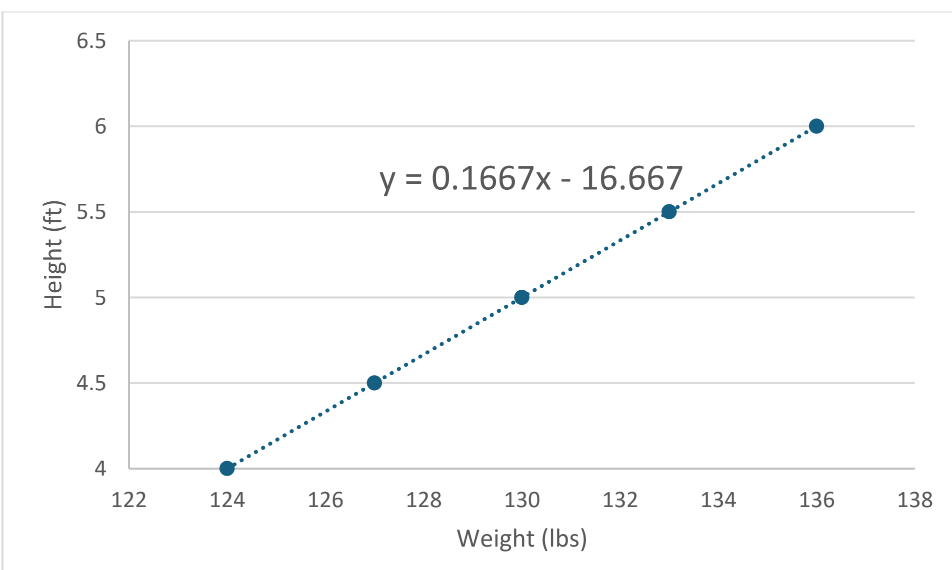
When discussing advanced topics like introduction to linear regression, such as in AP Statistics, going back to the relative nature of randomness will help introduce the probabilistic nature of the topic, which can sometimes be obscured when focusing entirely on its tedious, methodical procedures. We show how to introduce linear regression to include the concept of randomness as follows.

Teacher: Today we will introduce the topic of linear regression. Let us say we want to predict the height of people using their weights and suppose we have the following data. What can you say about this?



Lisa: It looks like a straight line.

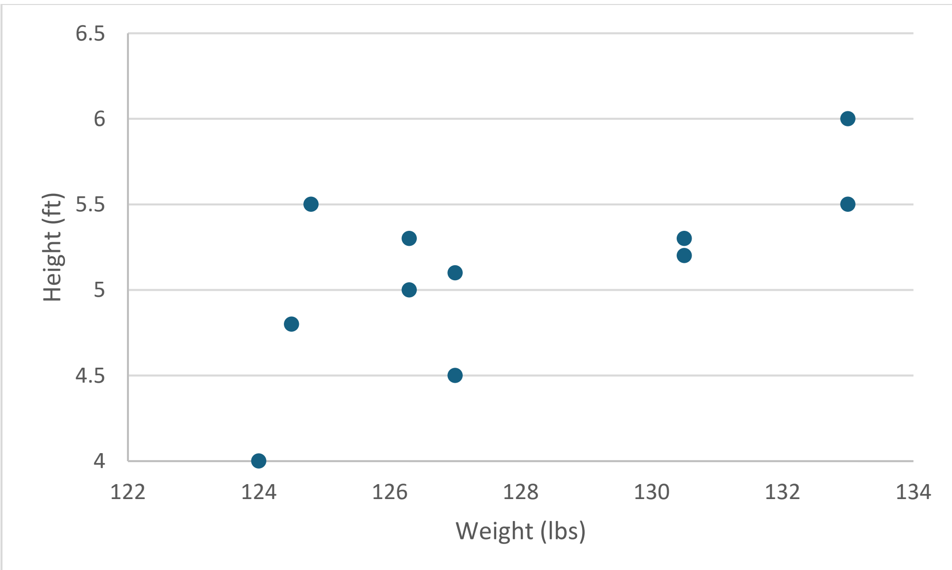
Teacher: It does, doesn't it? Thus, if we wanted to predict how tall someone who is 129 lbs is, we can draw a straight line that intersects each of those points as follows. From your knowledge of geometry and algebra, we know that knowing any two points of a line will let us obtain the equation of a line. Thus, we get the following equation as well.



Teacher: So we can simply substitute129 lbs for x, calculate, and get 4.833 ft. for y. However, this isn't very realistic at all now, is it? Why isn't it realistic? Joanna?

Joanna: Because different people who have the same weight can have different heights and vice versa?

Teacher: Exactly. So really, when we are trying to make this prediction, our data can look more like the following.



Teacher: So here, do you think the relationship between height and weight is exactly a straight line? (students shake their heads). You're right. It almost surely is not a straight line. But as we have brought up every now and then in our lessons, when we do statistics what do we need to make?

Students: Assumptions!

Teacher: Yes. We need to make the best assumptions that we can in the face of randomness. If we knew the exact biology that explains why someone with x weight has y height, then we would not need statistics, we can just do as we did in the previous example and compute, probably a much more complicated equation and then plug in the weight we want to interpolate. Instead, we don't know this, the process is random to us and we only see the data output. This data, what you see here, is no longer random because it already happened. So, the most we can do is think about the best model to assume about how this data is generated. One popular model which is what we will talk about today is the linear model…

From this point, the lesson can proceed as would a standard lesson on linear regression. The contribution of the relative nature of randomness is mainly on the rationale behind choosing a linear model. That is, we emphasize that this is a choice, not the truth. As George Box famously discussed, all models are wrong, some are useful (Box, 1976). Knowing this aids in understanding more advanced topics that students may encounter later in college or in practice, whereby more features are added to the linear model, such as clusterered standard errors, or non-linear models are preferred. Understanding that these are preferences made through expert consideration of what is known, rather than true representations of reality, helps immensely in making sense of why some models may be selected over others.

Clarifying Advanced Topics to Avoid Misconceptions
The validity of tools in statistical inference is hinged on randomness and do not function properly when used outside that context. When or if students ask if they are allowed to move the significance level from 0.05 to 0.10 after they see that their p-value is 0.06, an easy answer could be "no, because that's cheating." However, this answer suggests an ethical reason (which does exist) without explaining a technical one (i.e., Why is it cheating?). We demonstrate how to explain this better as follows.

Teacher: Does anyone have any questions about hypothesis testing? Ben?

Ben: If we get a p-value of 0.06 after setting alpha at 0.05, can't we just go back and set alpha to 0.10 instead?

Teacher: That's a great question. Remember that our hypothesis test is only interested in deciding if we can reject the null hypothesis based on the data. We set our alpha before we see the data. Why do we do this?

Ben: Because the test needs the data to be random?

Teacher: Exactly. And remember that randomness is relative to the observer. If we already see the data and in fact have computed the p-value and everything, what does that mean?

Ben: It is no longer random.

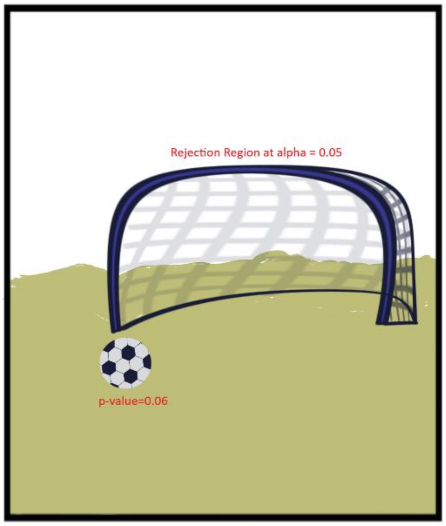
Teacher: Correct. So, if at that point, we decide to move alpha to 0.10, does this do anything?

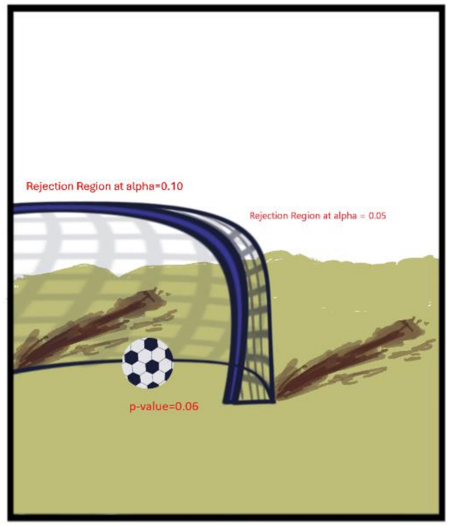
Ben: No, because it doesn't change the fact that we had set it to 0.05 before. It's like… making up the rules after you see the result so it suits what you want to see but doesn't change what you actually saw.

Teacher: Very good! Yes, Lisa?

Lisa: But then, would it work if we gathered new data on the same topic but this time set alpha to 0.10 from the beginning? It would, right?

Teacher: Excellent, you are right. If we endeavor to gather new data, that data would once again be random to us, which is why we can decide on whatever alpha we want for this new set prior to seeing it. Of course, we know that an alpha of 0.10 will give us a lower confidence about rejecting the null hypothesis than an alpha of 0.05 and that can be another, different reason why we cannot set it that high anyway, but the statistical analysis will remain valid unlike if we moved the goalpost after seeing the p-value.





Conclusion
Having a strong foundation in statistics is important not just for K-12 students who will seek to major in this field, but perhaps more importantly for all the other students who will seek other majors and for people in general who will encounter descriptive data like surveys or need to use tools for statistical inference for business, marketing, etc. The statistics majors will be exposed to this critical idea of relativity in randomness more than enough times in their major that we can count on it sticking eventually, whereas those who enter the other sciences will likely encounter just one or two more statistics courses and then eventually find themselves in situations where they think about why they are using a linear model instead of a model with quadratic terms, or why shouldn't they move their significance level to 0.10 after seeing that their p-value is 0.06? It would be excellent if at that moment they would recall your lessons on how randomness is relative and smile with the confidence of knowing better.

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