Lesson 9
Looking Forward with a Geometric Sequence and an Exponential Model

Kristin's Story – Chapter 5

Kristin recalled an interesting "word problem" posed by her middle school math teacher. She was not sure if it applied in her problem involving population projections, but she thought there might be a connection. Kristin rarely did her homework in her middle school days, and it bothered her mother. So, her teacher asked Kristin to consider making the following offer to her mother. If she did her homework today, she would receive 1 penny from her mother. If she did her homework the next day, she would receive 2 pennies. And, if she did her homework a 3rd day, she would receive 4 pennies. Could she persuade her mother that for every day she did homework, she would continue to receive double the number of pennies for at least 1 month? If her mother agreed to this arrangement, would Kristin also agree to do her homework every day?

Kristin was unable to convince her mother to agree to this plan as her mother did her own homework regarding this problem. But Kristin distinctly remembers that initially she was not sure it was a good plan. Her initial calculation indicated that after a full week of agreeing to this plan, she would receive $1.27. Hardly a major incentive to do homework for 7 days in a row. She played around with the numbers some more, and it started to strike her as more interesting. If she could stick it out for 14 days, the amount she would receive on the 14th day was $81.92, and the total amount paid to her for the 14 days was $163.83. It started to look worth the effort, but 14 days of homework was also a challenge. She worked with her math teacher and realized that what was happening was certainly worth her effort, but it also explained why her mother would not agree to this plan. They calculated that on the 28th day she would receive $1,342,177.28. The total amount received for the 28 days would be $2,684,354.55. Wow, how did that happen?

Kristin smiled over that memory. Would this same thinking have anything to do with population predictions?

Another model will be developed in this lesson based on a **geometric sequence**. Again, start off with the following estimates of the United States population from the US Census Bureau:

United States	2010	2015
Total count of people	309,348,193	320,896,618

Similar to your previous work with an arithmetic sequence, round off the above population estimates to the nearest millions of people, or:

United States

	2010	**2015**
Total count of people (in millions of people)	309	321

Lesson 9 - Problems

Geometric Sequence

A sequence in which an element is derived by multiplying a ***common ratio*** or constant factor to the prior element of the list is called a ***geometric sequence***. In the following problems, a proportion or common ratio is multiplied to each successive element resulting in a constant percent change in the population for each 5-year interval.

1. Consider the proportion $\frac{321}{309}$ or approximately 1.039 as the decimal representation to the nearest thousandth. What is the result of multiplying this proportion and 309 to the nearest million?

2. For the next estimate of the sequence, multiply 321 million by 1.039. Record your result to the nearest million in the population projection for 2020.

3. Continue multiplying each estimate by $\frac{321}{309}$ or 1.039 to obtain the next estimate until you reach the estimate for the year 2050. Complete the following table (round off your calculations to the nearest millions of people):

United States

Year	**2010**	**2015**	**2020**	**2025**	**2030**	**2035**	**2040**	**2045**	**2050**
Population (in millions of people)	309	321							
		x $\frac{321}{309}$ or 1.039	x $\frac{321}{309}$ or 1.039	x $\frac{321}{309}$ or 1.039	x $\frac{321}{309}$ or 1.039	x $\frac{321}{309}$ or 1.039	x $\frac{321}{309}$ or 1.039	x $\frac{321}{309}$ or 1.039	

4. The counts in the table form a *geometric sequence*. How would you describe the difference of an arithmetic sequence and a geometric sequence to someone unfamiliar with these sequences?

5. Consider a country in which the 2010 population was 309 million people and the 2015 population was also 309 million people. If a geometric sequence is generated to estimate the population in the future, what is the value of the common ratio or constant factor?

6. Consider a country in which the 2010 population was 200 million people and the 2015 population was 100 million people. If a geometric sequence is also generated for this country, what is the value of the common ratio or constant factor? How would you summarize the change in the population for each 5-year period?

7. By multiplying a population estimate by 1.039, what do you know about the next 5-year population estimate? For example, does the population grow, decline or stay the same? Derive the percent change in the population over each 5-year period as a percent of the population.

Exponential Model

The geometric sequence can be used to derive an exponential model to estimate the population for any given year. The exponential model is to the geometric sequence as the linear model was to the arithmetic sequence. The derivation of the exponential model, however, is more difficult to set-up. Consider the following exponential model derived from the above geometric sequence with modifications due to rounding off of key values:

$$y = 309(1.0078)^{x - 2010}$$

8. If the above exponential model is used to estimate the population for a given year, answer the following:
   a. Let x = 2010. What is the value of y?

   b. Let x = 2015. What is the value of y?

   c. What does x represent in the exponential model?

   d. What does 309 represent in the exponential model?

   e. What does y represent in the exponential model?

   f. What does 1.0078 represent?

9. Derive an estimate of the population for the start of the 2022 using the exponential model. Also derive an estimate of the population for the start of the year 2008. Would you have been able to derive these estimates using the values in the geometric sequence? Explain your answer.

10. The constant factor for the 5-year estimates of the geometric sequence was 1.039. The exponential model involves a constant factor of 1.0078. How do you think the constant factor of 1.0078 was derived?

11. Does the exponential model and the geometric sequence derive the same estimates for each 5-year interval? Complete the following table by calculating y from the exponential model. Let y represent an estimate of the population to the nearest millions of people. Recall that 309 million people at the start of 2010 was rounded off to the nearest millions of people from the Census data.

x	**2010**	**2015**	**2020**	**2025**	**2030**	**2035**	**2040**	**2045**	**2050**
y	309								

12. Compare the estimates from the exponential model to the estimates from the geometric sequence. Record population estimates from the model to the nearest million. Complete the following table:

Year	**Estimate of population from exponential model**	**Estimate of population based on the geometric sequence:**
2020		
2030		
2040		
2050		

13. The exponential model was derived from the first two values of the sequence. Why are the values in the table for problem 12 not exactly the same? Do you think the estimates are close? Explain your answer.

The International Data Base (IDB) of the Census Bureau indicates that the population for the United States in 1980 was 227,224,681 people, and the population for the United States in 2000 was 282,162,411 people. These counts were based on the census conducted in 1980 and the census conducted in 2000. They represent the best estimates of the actual counts of the United States population. For the following problems, round off each population to the nearest million, or the population in 1980 was 227 millions of people and the population in 2000 was 282 millions of people.

14. Compare the estimates derived from the exponential model to the estimates reported by the Census Bureau for 1980 and 2000. Is the exponential model close to the actual counts? Summarize your comparisons.

15. Do you think the estimates using the exponential model for years after 2015 will be accurate population projections for 2020 to 2050? Explain your answer.

Consider the 2010 and 2015 population counts for Kenya and Japan.

16. Complete the following geometric sequence for Kenya by multiplying each estimate by the common factor based on the 2010 and 2015 counts of people:

Kenya

Year	**2010**	**2015**	**2020**	**2025**	**2030**	**2035**	**2040**	**2045**	**2050**
Population (in millions of people)	41	46							

An exponential model to estimate the population of Kenya is:

$$y = 41(1.024)^{x - 2010}$$

17. Complete the following geometric sequence for Japan:

Japan

Year	2010	2015	2020	2025	2030	2035	2040	2045	2050
Population (in millions of people)	128	127							

An exponential model to estimate the population of Japan is:

$$y = 128(0.9984)^{x - 2010}$$

18. In what way is the exponential model for Japan different than the exponential models for the United States and Kenya?

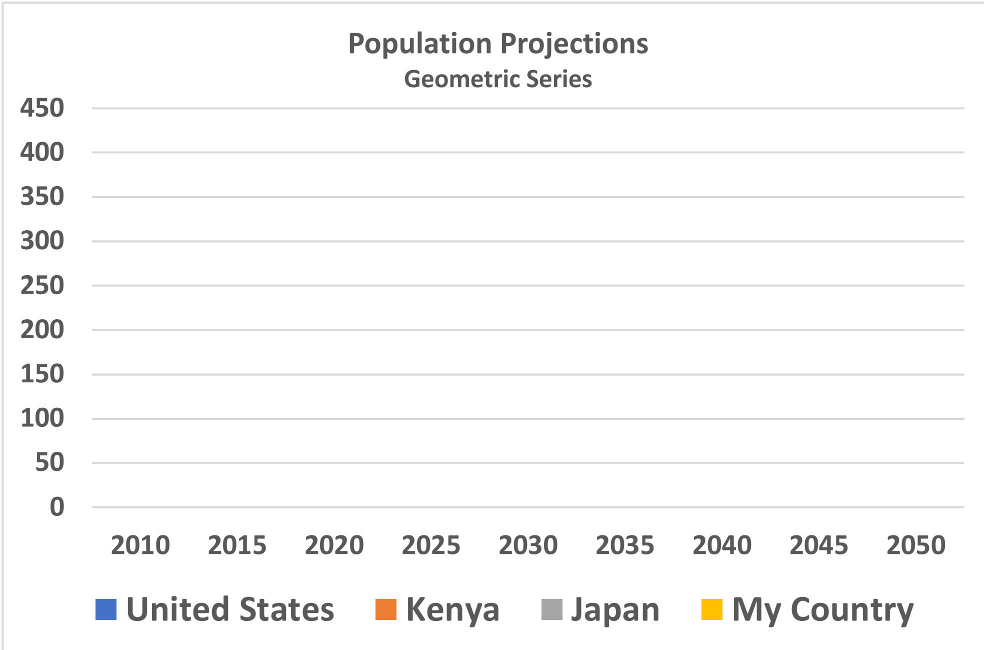
19. Finally, complete a geometric sequence for the population data you created for your own country, or the My Country data in Lesson 5.

My Country

Year	2010	2015	2020	2025	2030	2035	2040	2045	2050
Population (in millions of people)	163	157							

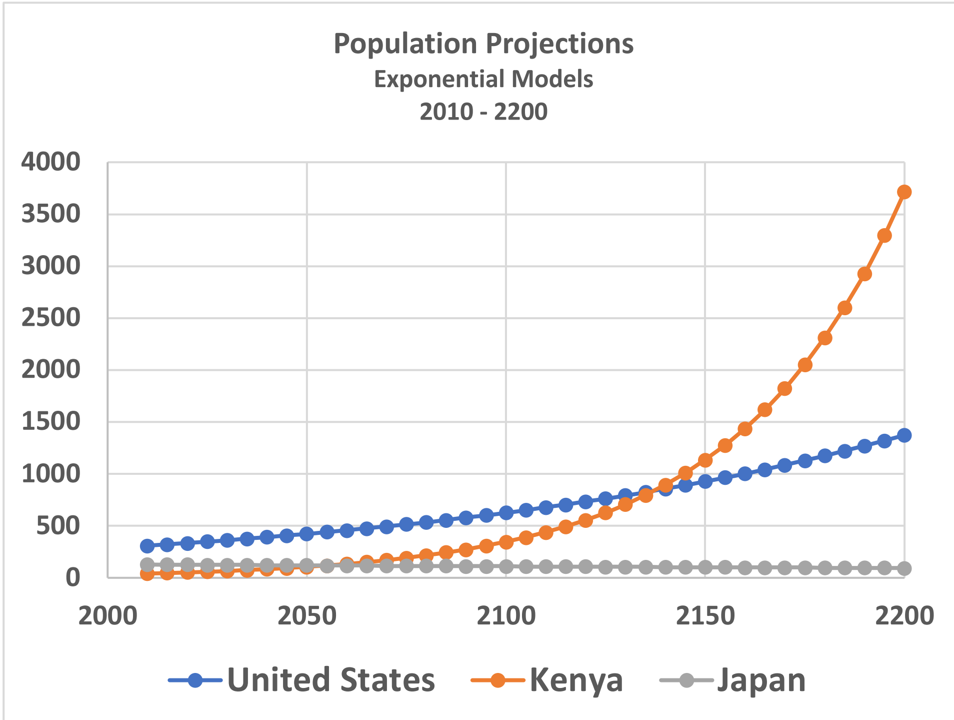
Consider deriving an exponential model for the above sequence of your country.

20. Use the values you derived for each geometric series of the United States, Kenya, Japan, and My Country data to graph their population projections.



Population in millions of people
(Use your own My Country results)

After all is said and done, how do we know if the projections from any model are on target? Some of the ways to check a model is to readjust it over time. The models derived in this lesson and the last lesson might be modified by the results of a 2020 census. Clearly the arithmetic model and the exponential model cannot be used forever. Study the following graph that summarizes the exponential models' projections for the United States, Kenya, and Japan from 2010 to 2200. What is happening to the population in the United States? Kenya? Japan?



Population in millions of people

Without some adjustments, the above models predict that a country's population in the future will be either unreasonably large or near extinction. What could alter the projections of these models? What events in the previous lessons of **Unit 2: *Looking Back*** might impact the future of the United States, Kenya, and Japan and alter these forecasts?

Recap the exponential model by recording the 2050 estimates to the nearest millions of people for each country . Compare these estimates to the United States Census estimates.

Projections of the 2050 population projects:

	Census Bureau model	Linear model	Exponential model
United States	398,328,349	405 million people	
Kenya	70,755,460	81 million people	
Japan	107,209,536	120 million people	

Similar to the linear models developed in Lesson 8, the exponential models derived in this lesson provide estimates of the ***total*** population for the United States or Kenya, or Japan. In their current form, the models do not provide estimates of the age groups that make-up a population.