Teaching Notes

Lesson 8

Looking Forward with an Arithmetic Sequence and a Linear Model

Overview:

This lesson is the first in a series of lessons that explore models that estimate a country's population in the future. The problems in this lesson ask students to think about what a population projection is, what is a model that results in a population projection, why is this important, and are these projections reliable. Answers to these problems are examined by developing projection models, analyzing the projections from the models, and evaluating if the models provide accurate estimates of past population distributions. Essentially, answers to these problems are grounded in applying and interpreting mathematical tools.

This lesson begins with a basic linear model using the 2010 and 2015 population totals. The conclusions are limited as this model is not developed to provide projections of the main discussion points of this module, namely the counts in the ***age groups*** that make-up a country's population.

This lesson stretches students' development of the **Modeling Continuum** by moving their thinking to levels 3 and 4. Students are involved in incorporating data into the projection models and making "What if …?"decisions. An alignment of the problems in this lesson to the **Modeling Continuum** are suggested in the following table:

Modeling Continuum Classification

Level 1	Level 2	Level 3	Level 4
Problems: 1	Problems: 2, 3, 4, 5, 8	Problems: 6, 7, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21	Problems: 9, 10

Primary tools students use in this lesson to answer the above problems are:
Arithmetic operations, proportions, percent, extracting and interpreting data from graphs, working with coordinate graphs, slopes, and solving and setting up linear equations.
See the connection of these tools to high school standards in the ***Overview of the Module***.

Resources needed for this lesson:

Provide a copy of the **Introductio**n to Unit 3 and a complete Lesson 8 for each student. This lesson does not require any additional handouts.

Launch:
Direct students to read the introduction to Unit 3. Consider asking students: "If you and your family remain healthy and well, how old will you be in 2050? How old will your parents or guardians be in 2050? How old will your teacher be in 2050?" Looking forward, or looking into the future, is the goal of this lesson.

Ask students why looking forward is important? Expect mixed responses to that question, including that it is not an important question. Discuss, however, the implications for a country if its population increases by millions of people in the next several decades. What if those increases were not anticipated? Also discuss the implications for a country if its population decreases by millions of people. Emphasize that discussing the future includes them in the count of people, and that they will see their country through the eyes of possibly a different shape as defined in previous lessons. After these opening questions and discussion, direct students to complete the problems.

Implementation Ideas:
This lesson involves some topics that may be familiar to some students and new to others. Topics such as what is an arithmetic sequence, the slope of a linear equation, the y-intercept of a linear equation, and the best-fitting line are topics most students have studied, but they may be topics that not all students understand or readily apply. Modify the expectations based on students' previous understanding of sequences and linear equations if these topics are unfamiliar to students. The depth of explanations of the topics involving slope, y-intercept, best-fitting line, and scatter plot are minimal and may require additional explanations. The problems in this lesson, however, can be used to strengthen an understanding of these topics, as well as provide connections to meaningful applications.

This lesson is the first of several lessons that introduce students to specific ***models*** that look into the future. The projections in this lesson are derived from equations based on population estimates for 2010 and 2015 of the United States, Kenya, and Japan. Discuss with students what might alter some of the estimates that are derived from these models. For example, will the count of people moving into a country each year be the same? Will a country record a similar count of births each year? If those counts are not the same, how will the models for each country be altered? Begin discussing these questions as students work through the problems in Lesson 8 and in the next several lessons.

Student responses or descriptions
Lesson 8 - Problems

Arithmetic Sequence
1. Calculate the difference of the 2015 and 2010 population estimates.
   12 million people

2. Add the above difference to the 2015 population. The result is an estimate for the 2020 population assuming the population increases by the same count from 2015 to 2020 as it did from 2010 to 2015. In the same way, add the difference to the 2020 estimate to obtain the 2025 population estimate. Complete the following table by continuing the process:

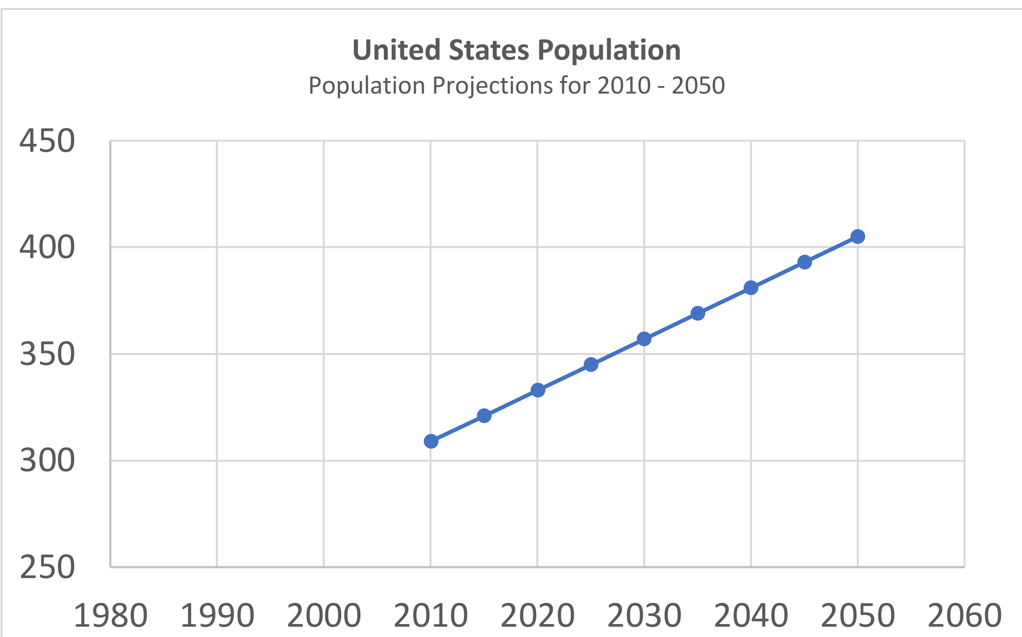
United States

Year	**2010**	**2015**	**2020**	**2025**	**2030**	**2035**	**2040**	**2045**	**2050**
Population (in millions of people)	309	321 *+12*	*333* *+12*	*345* *+12*	*357* *+12*	*369* *+12*	*381* *+12*	*393* *+12*	*405*

3. The above estimates form a list that is described as a ***finite arithmetic sequence***. Based on this example, how would you explain an arithmetic sequence to someone who was unfamiliar with it?
The constant difference between the numbers that started the sequence is added to each element to derive the next element. The sequence is a list of numbers where each successive term of the sequence is found by adding a constant.

Linear Model

4. Consider the following coordinate grid in which the x-axis represents the year and the y-axis represents the estimated population values. Plot each of the coordinate points of the sequence in problem 3 on this grid or a grid of this type. The first two points are plotted for you.
The following graph connects the points based on the arithmetic sequence.



Population in millions of people

5. Connect the coordinate points you plotted on the graph. What do you notice?
   The points appear to lie on a straight line when connected.

6. There are several procedures designed to derive an equation given two points. Consider the following steps to derive a linear equation by representing the coordinate values (x , y) for 2010 as (x_1, y_1) and 2015 as (x_2, y_2).
   a. Calculate the slope or change in population per year (to the nearest tenth) using the two points (2010, 309) and (2015, 321) as represented below:

      Slope = $\frac{y2-y1}{x2-x1} = \frac{Change\ in\ population}{change\ in\ years} = \frac{321-309}{2015-2010} = \frac{12}{5}$ *or 2.4 million people per year.*

      Explain this slope as a change in the population in 1 year.
      The country is projected to grow by a count of 2.4 million people per year.

   b. An equation called the "***point-slope***" equation derives the slope based on the two given points. The slope is then used with one of the two points to create a linear model. For this problem, let x represent the year and y represent the population in millions. Use the slope previously calculated from the two points (2010, 309) and (2015, 321). Use the first point (2010, 309) to complete the equation of the linear model. The steps are outlined below:

      $y - y_1 = \frac{y2-y1}{x2-x1}(x - x_1)$ for (x_1, y_1)

      Use the point (2010, 309) and the value of the slope to complete this linear model:
      y - ____ = $\frac{(y2-y1)}{(x2-x1)}$ (x - ____)

      Using (2010, 309) as the coordinate point and the value of the slope, the equation becomes:
      y – 309 = 2.4(x – 2010)

   c. If an estimate of the population for 2020 is derived from this equation, what is the value of x used in this model ? Derive an estimate of the population for 2020 using the above. Round your estimate of the 2020 population to the nearest millions of people.
      The value of x would be the year 2020, and the value of y would be the projected population estimate for 2020.
      y – 309 = 2.4(2020 – 2010)
      y – 309 = 2.4(10)
      y – 309 = 24
      y = 24 + 309 = 333

The estimated population of the United States in 2020 would be 333 million people.

d. Rework the above equation into a form called the "***slope - y-intercept***" linear equation. This form is also a representation addressed in a study of algebra. This form is often summarized as **y = mx + b** where **m** is the slope previously calculated in the above problem and **b** is the y-intercept, or the value of y when x =0. (There are, however, other representations of the ***slope - y-intercept*** equation. For example, a statistical representation of this equation is often summarized as $y = b_0 + b_1x$ were b_1 represents the slope and b_0 is the y-intercept.)

$y - y_1 = 2.4(x - x_1)$

$y - 309 = 2.4(x - 2010)$

$y - 309 = 2.4x - 4824$

$y = 2.4x - 4824 + 309$

$y = 2.4x - 4515$

7. Answer the following.
   a. Determine the ***units*** to complete the following statement:
   The slope is "__________" per "_______"
   The slope is 2.4 million people per year, or people per year.
   b. What does a positive slope indicate?
   The population is increasing constantly per year.
   c. What would a negative slope indicate?
   The population is decreasing constantly per year.
   d. What does a slope that is equal to 0 indicate?
   The population is neither increasing nor decreasing per year. It stays the same.

8. The United States Census estimates for 1980, 1990, 2000, and 2005 are included in the following table. Use the linear model to derive population estimates for each of these years. Record your estimates in the table.

	1980	1990	2000	2005
US Census estimates	227	250	282	296
Linear model estimates	*237*	*261*	*285*	*297*

9. Compare the estimates derived from the linear model to the counts of people reported by the Census Bureau.
   a. Calculate the difference between the linear model estimate and the Census Bureau for 2005. Do you think the linear model provided a good estimate for 2005? Explain your answer.
   The difference was 1 million people. This is very close.

b. In the same way, calculate the differences between linear model estimates and the estimates reported by the Census Bureau for 2000? 1990? 1980? Do you think the linear model provided a good estimate for those years? Explain your answer.
Direct students to answer the years one at a time. The difference in 2000 is 3 million people. Consider this difference as also a good estimate. The difference in 1990 is 11 million people. This estimate is not as good. The difference in 1980 is 10 million. This estimate is also not as good.

c. Derive the estimates for the years 2010 and 2015 using the linear model. Are the estimates different from the previous estimates? Explain your answer.
$y = 2.4(2010) – 4515$
$y = 309$
$y = 2.4(2015) – 4515$
$y = 321$
The estimates are equal to the Census estimates as each year was used in constructing the model.

10. The Census Bureau estimates that the population of the United States at the time of the signing of the Declaration of Independence in 1776 was approximately 2.5 millions of people. This estimate is not based on any census as the United States Constitution was not completed at that time. What is the linear model's estimate of the population in 1776? Do you think the linear model provided a good estimate for 1776? Explain your answer.
The linear model estimate is: $y = 2.4(1776) – 4515$, or $y = - 252.6$
Obviously this estimate is extremely incorrect as a negative population does not make sense.

11. Estimate the following future counts of the United States using the linear model:
    a. 2020 (332)
    $y = 2.4(2020) – 4515 = 333$ millions of people
    b. 2030 (343)
    $y = 2.4(2030) – 4515 = 357$ millions of people
    c. 2050 (397)
    $y = 2.4(2050) – 4515 = 405$ millions of people
    d. 2300 (947)
    $y = 2.4(2300) – 4515 = 1005$ millions of people

12. What years do you think the estimates derived in problem 11 are accurate predictions of the future? What years do you think the estimates derived in problem 11 are not accurate predictions of the future? Why would an estimate from a model like the linear model not be a good estimate?
The best estimates are probably for 2020 and 2030 as they are close to the years used to derive the model. The estimates for 2050 and 2300 are probably not accurate and could be quite different than what will be the actual population.

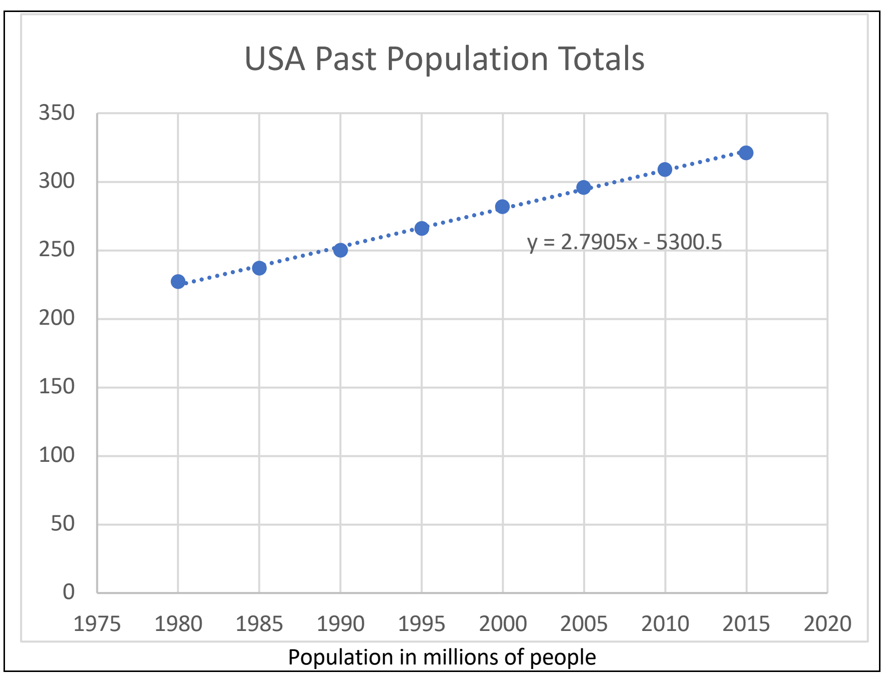
The model begins to fail as changes in a 5-year period will be different than the changes used in deriving the model. Changes due immigration, emigration, births, and deaths are not consistent.

There are many other models that could be developed, including other linear models. Consider the data collected from the Census Bureau of the United States population from 1980 to 2015 and a scatterplot based on this data. Each of these counts are considered the best estimates of the actual population.

Year	**1980**	**1985**	**1990**	**1995**	**2000**	**2005**	**2010**	**2015**
Population (in millions of people)	227	237	250	266	282	296	309	321

13. Sketch a line that you think is a best-fitting line of the scatter plot. Identify any two points on the best-fitting line you drew and derive a linear equation based on the two points you selected using the point-slope summary. (The points you select will probably not be points represented by the scatter plot.) Determine the estimates of the United States population for the following years using the equation of the best-fitting line. Estimate each population to the nearest millions of people:

Year	**2020**	**2025**	**2030**	**2035**	**2040**	**2045**	**2050**
Population estimates (in millions of people)							



Answers will vary given the line drawn by students. If the line approximates a best-fitting line, work with students to estimate the population estimates using their line. Encourage students to derive the slope, y-intercept, and the resulting equation of the line they drew by estimating the values from two points on the line (and remind them that the points do not necessarily have to be the data points). They will then use the equation to estimate the values for the years in the table.

If technology is available and similar to a graphing calculator with software, the data provided for the scatter plot can be used to derive the linear regression equation of the scatter plot. The linear regression equation or least squares regression equation is approximately equal to the following:

$$y = 2.8x - 5300$$

The following projected estimates are based on the least square regression equation:

Year	***2020***	***2025***	***2030***	***2035***	***2040***	***2045***	***2050***
Population (in millions of people)	*356*	*370*	*384*	*398*	*412*	*426*	*440*

Note that the slope of this line is greater than the slope of the linear model in problem 6, and the population estimates are considerably higher over time based on this model.

14. Compare the estimates derived from the linear model in problem 6 to the estimates derived from the best-fitting line.
The values to complete this table will vary based on the estimate of the best-fitting line equation. The values in the following table were based on the least squares' regression equation.

Year	**2020**	**2025**	**2030**	**2035**	**2040**	**2045**	**2050**
Population estimates from model in problem 6 (in millions of people)	*333*	*345*	*357*	*369*	*381*	*393*	*405*
Population estimates from best-fitting line (in millions of people)	*356*	*370*	*384*	*398*	*412*	*426*	*440*

Do the models produce similar estimates? Explain.
The models are different, with the best-fitting model projecting higher estimates of the population in the future. Discuss with students that differences this large may result in

varied decisions regarding the building of schools, infrastructure (roads, airports, etc.), financial planning, and many other issues frequently discussed and debated.

15. Complete the following table for Kenya by adding the difference of the 2015 population and the 2010 population to obtain the next population estimates.

Kenya

Year	**2010**	**2015**	**2020**	**2025**	**2030**	**2035**	**2040**	**2045**	**2050**
Population (in millions of people)	41	46	*51*	*56*	*61*	*66*	*71*	*76*	*81*

16. Derive a linear model for the above population estimates for Kenya in the same way as linear model was developed for the United States in problem 6.
The linear equation based on the above sequence is:
$y = x - 1969$
Note: the change in the population is estimated to be approximately 1 million people per year.

17. Complete the following table for Japan by adding the difference of the 2015 population and the 2010 population to obtain the next population estimates.

Japan

Year	**2010**	**2015**	**2020**	**2025**	**2030**	**2035**	**2040**	**2045**	**2050**
Population (in millions of people)	128	127	*126*	*125*	*124*	*123*	*122*	*121*	*120*

18. Derive a linear model for the above population estimates for Japan in the same way as linear model was developed for the United States in problem 6.
The linear equation based on the above sequence is:
$y = -0.2x + 530$
Note: the change in the population is estimated to decrease by approximately 200,000 people per year.

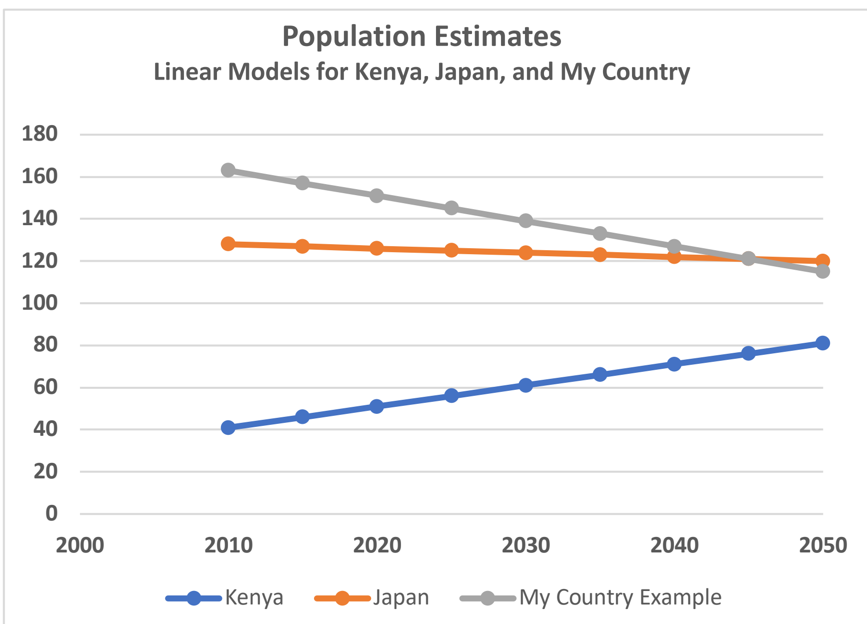
19. Finally, complete a table for the population you created for your own country, or My Country.
The values entered in the table will vary depending on the My Country population totals designed by the students in previous lessons. If students completed their own country data, they should be encouraged to use the data and derive a linear model in the same way as the previous examples. The values entered in the following table were provided in the Teaching Notes to analyze a country whose shape is classified as a top-layered country.

My Country

Year	2010	2015	2020	2025	2030	2035	2040	2045	2050
Population (in millions of people)	163	157	*151*	*145*	*139*	*133*	*127*	*121*	*115*

20. Derive a linear model for the population estimates of My Country in the same way as the linear model was developed for the United States.
Answers will vary depending on the students' data. Using the example provided in the above table, the linear equation is:
y = - 1.2x +2575

21. Use the following grid to record the projected population values for Kenya, Japan, and My Country based on the linear models developed in this lesson
The following sketch includes the points and line for the example of My Country previously discussed.



Population in millions of people

The linear models developed in this lesson provide estimates of the total population. The models do not, however, provide estimates of the age groups that make-up the population. This challenge must still be addressed.

Assessment Ideas:

Assessment Task:

Consider the following assessment task to determine a student's understanding of the lesson.

Paul used population data of his fictitious country from Lesson 5 and derived a linear model of $y = 2.5x - 2000$, where x represented the year and y represented the count of people for that year in millions of people.

a. If Paul's model is accurate, how many people are added to his country's population each ten years?

b. Paul derived his model using the 2010 and 2015 population totals for his country. Do you think his model will be accurate for predicting the population in 2100? Explain your answer.

c. Paul also was interested in the estimated population of his country in 2005. Do you think his model will be accurate in predicting the population in 2005? Explain you answer.

Comments on the Assessment Task:
The assessment task is a general review of the main goals of this lesson. Paul's country will grow by 2.5 million people per year which is based on an understanding of slope. In ten years, the country will grow by 25 million people (provided the linear model is accurate). Paul would not consider the 2100 estimate as accurate as it is several decades away from the years used to derive the model. However, Paul would consider the estimate for 2005 to be accurate as it is close to the years used to derive the model.

Additional Assessment Ideas:
Ask students what factors would alter the population projections using the linear models as a wrap-up question. What could happen to a country during a 5 or 10-year period that would result in a count of the total population that is different than the count projected by the linear model? This question provides opportunities for summarizing this lesson, and also suggesting where the next several lessons are headed.

Consider directing students complete an **Exit Summary** for this lesson. Discuss with students the levels and in particular, levels 3 and 4 as more questions and problems require use of the tools to make more independent summaries or decisions.