Teaching Notes
Lesson 9
Looking Forward with a Geometric Sequence and an Exponential Model

Overview:

This lesson works with students in deriving a geometric sequence based on the 2010 and 2015 population estimates. From this sequence, an exponential model is presented and used to derive estimates of the count of people in the future. Similar to the last lesson, however, the conclusions are limited as the model is not designed to provide counts in the age groups that make-up a country's population. The role of an exponential projection model is another tool in estimating the total population of a country over time.

The exponential model represents the most robust model of the projection models. This lesson concludes with the observation that if this model were to continue indefinitely, populations of growing countries will ultimately be too large to support the population, and populations of declining countries will ultimately approach 0. This observation continues to provide discussion about other models and their impact over time.

This lesson stretches students' development of the **Modeling Continuum** by moving their thinking to solve problems to levels 3 and 4. Students are involved in incorporating data into the projection models and making "What if …?"decisions. An alignment of the problems in this lesson to the **Modeling Continuum** are suggested in the following table:

Modeling Continuum Classification

Level 1	Level 2	Level 3	Level 4
Problems: 1, 2	Problems: 3, 4, 12, 15, 16, 19, 20	Problems: 5, 6, 7, 8, 9, 10, 11, 13, 17, 18	Problems: 14, 15

Primary tools students use in this lesson to answer the above problems are:
Arithmetic operations, proportions, percent, extracting and interpreting data from graphs, working with coordinate graphs, and solving and setting up an exponential model or equation. See the connection of these tools to high school standards in the ***Overview of the Module***.

Resources needed for this lesson:

Provide a copy of a complete Lesson 9 for each student. This lesson does not require any additional handouts.

Launch:

Launch the lesson by providing time for students to read ***Kristin's Story – Chapter 5***. Discuss some of the details of the doubling results highlighted in the story. This story provides an example of a geometric sequence (and the resulting series) in which the common factor multiplied to derive the next payment for doing her homework is 2. Write out the sequence detailed in the story and discuss how the values cited in the story were derived. Consider providing other examples such as the amount of money earned in a compound savings' account.

Pose a few questions before students start the problems, and then revisit these discussion questions at the end of the lesson as a wrap-up:

- What do you think will happen to a country's population if it increases its population every year for over 200 years?
- Why would a country grow every year? Why would a country not grow every year?
- Is it possible for a country to grow for a period of time and then not grow for a similar period of time? Think of specific examples.

Implementation Ideas:

The problems are designed for students to complete individually or in small groups. They are also designed with some expectation that students have previously worked with an exponential function, although if they did not, the problems allow you to introduce and expand their understanding of an exponential model. Details involved in setting up the exponential model are minimal. Students learn in this lesson the starting value and the common ratio or common factor of an exponential function. The problems are connected to the goals of this module involving the count of people in a country. However, if time is available, financial problems involving compound interest, mortgages, future and present value are related topics that have a foundation in the exponential model.

Provide time for students to complete the problems, with periodic whole class discussions of their answers. The explanations provided in the Teaching Notes include a summary of the answers to several problems or explanations of the problems that students are expected to develop. Several of the problems also provide an opportunity for you to check their understanding of this important model.

Student responses and descriptions
Lesson 9 - Problems
Geometric Sequence

1. Consider the proportion $\frac{321}{309}$ or approximately 1.039 as the decimal representation to the nearest thousandth. What is the result of multiplying this proportion and 309 to the nearest million?
   $\frac{321}{309}$ x 309 million = 321 million

2. For the next estimate of the sequence, multiply 321 million by 1.039. Record your result to the nearest million in the population projection for 2020.
   321 x 1.039 million = 334 million

3. Continue multiplying each estimate by $\frac{321}{309}$ or 1.039 to obtain the next estimate until you reach the estimate for the year 2050. Complete the following table (round off your calculations to the nearest millions of people):

United States

Year	**2010**	**2015**	**2020**	**2025**	**2030**	**2035**	**2040**	**2045**	**2050**
Population (in millions of people)	309	321	*334*	*347*	*361*	*375*	*390*	*405*	*421*
		x $\frac{321}{309}$ or 1.039	x $\frac{321}{309}$ or 1.039	x $\frac{321}{309}$ or 1.039	x $\frac{321}{309}$ or 1.039	x $\frac{321}{309}$ or 1.039	x $\frac{321}{309}$ or 1.039	x $\frac{321}{309}$ or 1.039	

4. The counts in the table form a *geometric sequence*. How would you describe the difference of an arithmetic sequence and a geometric sequence to someone unfamiliar with these sequences?
   *Each next count in the list of a geometric sequence is derived by **multiplying** the previous count by a constant factor . In an arithmetic sequence each count is derived by **adding** (or subtracting) a constant.*

5. Consider a country in which the 2010 population was 309 million people and the 2015 population was also 309 million people. If a geometric sequence is generated to estimate the population in the future, what is the value of the common ratio or constant factor?
   The constant factor would be equal to 1. A factor of 1 indicates the population stays the same for each 5-year period.

6. Consider a country in which the 2010 population was 200 million people and the 2015 population was 100 million people. If a geometric sequence is also generated for this country, what is the value of the common ratio or constant factor? How would you summarize the change in the population for each 5-year period?
   The population factor would be equal to one-half, or 0.5. The population would be cut in half for each 5-year period.

7. By multiplying a population estimate by 1.039, what do you know about the next 5-year population estimate? For example, does the population grow, decline or stay the same? Derive the percent change in the population over each 5-year period as a percent of the population.
   The population will increase for each 5-year period. The percent increase for each 5-year period will be 3.9%. .

Exponential Model

The geometric sequence can be used to derive an exponential model to estimate the population for any given year. Consider the following exponential model derived from the above geometric sequence with modifications due to rounding off key values:

$$y = 309(1.0078)^{x - 2010}$$

8. If the above exponential model is used to estimate the population for a given year, answer the following:
   a. Let x = 2010. What is the value of y?
      The value of y would be 309 to the nearest whole number.
   b. Let x = 2015. What is the value of y?
      The value of y to the nearest whole number is 321 to the nearest whole number.
   c. What does x represent in the exponential model?
      x represents the independent variable and the year when calculating the population.
   d. What does 309 represent in the exponential model?
      309 is the starting value in the geometric series, or the population at the start of the given values.
   e. What does y represent in the exponential model?
      y represents the dependent variable or the value of the population to the nearest whole number.
   f. What does 1.0078 represent?
      1.0078 represents the common factor of the geometric series or the exponential model.

9. Derive an estimate of the population for the start of the 2022 using the exponential model. Also derive an estimate of the population for the start of the year 2008. Would you have been able to derive these estimates using the values in the geometric sequence? Explain your answer?

   For 2022, let x = 2022

   $y = 309(1.0078)^{2022-2010}$ *or* $y = 309(1.0078)^{12}$

   $= 339$ *million people*

   For 2008, let x = 2008

   $y = 309(1.0078)^{2008-2010} = 309(1.0078)^{-2} = 304$ *million people*

   The above calculations could be estimated from the sequence by doing a messy extrapolation of the values between 2020 and 2025. It is more straightforward to use the exponential model. Allow students to indicate that they would not have been able to derive these estimates from the geometric sequence.

10. The constant factor for the 5-year estimates of the geometric sequence was 1.039. The exponential model involves a constant factor of 1.0078. How do you think the constant factor of 1.0078 was derived?

    Students could explain that the constant factor is an estimate of a yearly factor and would be 1/5 of the constant factor for a 5-year period, or 1/5 of 0.039. This is approximately 0.0078. An exponential model, however, does not exactly work like this, therefore, another method to solve for r (where r is the yearly constant factor) is indicated below. This method is addressed in an intermediate or advance algebra class.

    $321 = 309r^5 \Rightarrow 321/209 = r^5 \Rightarrow \sqrt[5]{\left(\frac{321}{309}\right)} = r \Rightarrow$ *r is approximately 1.00765.*

    The resulting yearly common factors are approximately the same for each method.

11. Does the exponential model and the geometric sequence derive the same estimates for each 5-year interval? Complete the following table by calculating y from the exponential model. Let y represent an estimate of the population to the nearest millions of people. Recall that 309 million people at the start of 2010 was rounded off to the nearest millions of people from the Census data.

x	**2010**	**2015**	**2020**	**2025**	**2030**	**2035**	**2040**	**2045**	**2050**
y	309	*321*	*334*	*348*	*362*	*376*	*391*	*407*	*423*

12. Compare the estimates from the exponential model to the estimates from the geometric sequence. Record population estimates from the model to the nearest million. Complete the following table:

Year	Estimate of population from exponential model	Estimate of population based on the geometric sequence:
2020	*334 million*	*334 million*
2030	*362 million*	*361 million*
2040	*391 million*	*390 million*
2050	*423 million*	*421 million*

13. The exponential model was derived from the first two values of the sequence. Why are the values in the table for problem 12 not exactly the same? Do you think the estimates are close? Explain your answer.
Differences are due to the round off of the common factor and the population estimates. The estimates are very close.

14. Compare the estimates derived from the exponential model to the estimates reported by the Census Bureau for 1980 and 2000. Is the exponential model close to the actual counts? Summarize your comparisons.
Population is represented in millions of people:

Year	*Exponential Estimates*	*Population from Census Bureau (best estimate of actual population)*
1980	*245 million*	*227 million*
2000	*286 million*	*282 million*

15. Do you think the estimates using the exponential model for years after 2015 will be accurate population projections for 2020 to 2050? Explain your answer.
Point out to students that when looking back the estimate for 1980 was noticeably different than the best estimate of the actual population from the Census Bureau, however, the estimate for 2000 was considerably closer to the Census estimate. Answers will vary but discuss with students that population estimates closer to the starting population estimates for 2010 and 2015 are anticipated to be closer to what will be (or what was) the actual population estimates. This was the same observation students made with the arithmetic sequence or the linear model..

Consider the 2010 and 2015 population counts for Kenya and Japan.

16. Complete the following geometric sequence for Kenya by multiplying each estimate by the common factor based on the 2010 and 2015 counts of people:

Kenya

Year	**2010**	**2015**	**2020**	**2025**	**2030**	**2035**	**2040**	**2045**	**2050**
Population (in millions of people)	41	46 x 1.12	*52 x 1.12*	*58 x 1.12*	*65 x 1.12*	*73 x 1.12*	*82 x 1.12*	*92 x 1.12*	*103*

To derive the next count in the list in millions, multiply the previous count by $\frac{46}{41}$ or approximately 1.12. This factor indicates an increase in the population of approximately 12% for each 5-year period.

17. Complete the following geometric sequence for Japan:

Japan

Year	**2010**	**2015**	**2020**	**2025**	**2030**	**2035**	**2040**	**2045**	**2050**
Population (in millions of people)	128	127 x 0.992	126 x 0.992	125 x 0.992	124 x 0.992	123 x 0.992	122 x 0.992	121 x 0.992	120

To derive the next count in the list in millions, multiply the count by $\frac{127}{128}$ or approximately 0.992 This common factor indicates an approximately 0.8% decrease in each 5-year period.

Note: Discuss with students why the decrease in Japan's population looks consistent (or, a decrease of 1 million people for each 5-year period). It almost looks like a linear model. The constant factor is close to 1. As a result, the decrease is slight for each year, but accumulates to approximately 0.8% for each 5-year period. If, however, we project the populations far into the future, the decreases are more noticeable in comparison to the starting values.

18. In what way is the exponential model for Japan different than the exponential models for the United States and Kenya?

The population of Japan is forecast to decrease. Discuss with students that for a decreasing geometric sequence, the constant factor is equal to (1 – the rate of change per year), as opposed to (1 + the rate of change per year) which is the case for Kenya and the United States. If students have previously worked with compound interest problems, including present value or future value problems , they may have worked with formulas that included (1 + i) or (1 – i) where "i" is the interest rate. The above examples are similar.

19. Finally, complete a geometric sequence for the population data you created for your own country, or the My Country data in Lesson 5.

Students complete the table using the data they created for My Country. The data in the following table is based on the fictitious My Country introduced in the Teachingr Notes of Lesson 5. It continues to provide an example of what happens to a top-layered country. The data and the resulting model are intended to provide an opportunity to discuss how a model affects different types of countries.

My Country

Year	**2010**	**2015**	**2020**	**2025**	**2030**	**2035**	**2040**	**2045**	**2050**
Population (in millions of people)	163	157 x 0.963	151 x 0.963	145 x 0.963	139 x 0.963	134 x 0.963	129 x 0.963	124 x 0.963	119

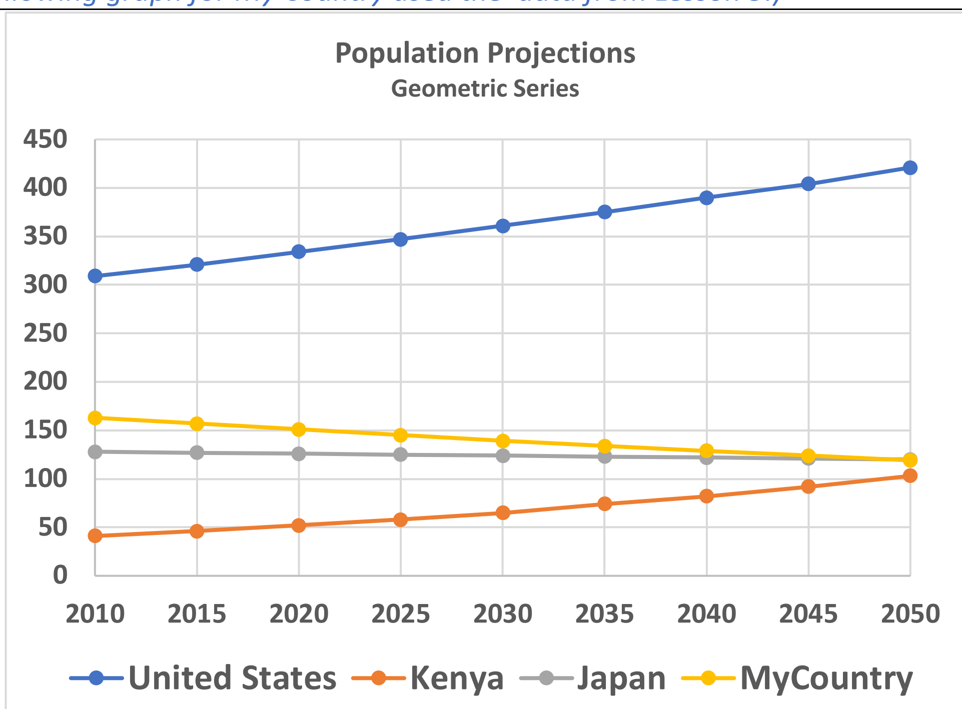
The constant factor used in the example of My Country is $\frac{157}{163}$ *or approximately 0.9631 which is approximately a decrease of 0.0368 or 3.68% per 5-year period. A yearly estimate of 0.0074 or a decrease of 0.74% per year is used in the following model.*

An exponential mode for the above is:

$$y = 163(1 - 0.0074)^{x-2010} \text{ or } y = 163(0.9926)^{x-2010}$$

20. Use the values you derived for each geometric series of the United States, Kenya, Japan, and My Country data to graph their population projections.

(The following graph for My Country used the data from Lesson 5.)



Population in millions of people
My Country example – students use their own country data

Recap the exponential model by recording the 2050 estimates to the nearest millions of people for each country . Compare these estimates to the United States Census estimates.

Projections of the 2050 population projects:

	Census Bureau model	Linear model	Exponential model
United States	398,328,349	405 million people	*421 million people*
Kenya	70,755,460	81 million people	*103 million people*
Japan	107,209,536	120 million people	*120 million people*

Similar to the linear models developed in Lesson 8, the exponential models derived in this lesson provide estimates of the total population for the United States, or Kenya, or Japan. In this current form, the models do not provide estimates of the age groups that make-up a population.

Assessment ideas:

Assessment Task:

Consider the following assessment task to determine a student's understanding of the lesson.

Oostburg is a village in Wisconsin that had a population of 1000 people in 2010 and a population of 1050 in 2015. In addition, Milwaukee, Wisconsin had a population of 600,000 in 2010. Milwaukee's population declined 4% from 2010 to 2015. Changes in the population are anticipated to continue for the next several decades for both Oostburg and Milwaukee. If geometric sequences are derived for both Oostburg and Milwaukee to estimate their population counts for 5-year periods, determine the following:

a. What is the common ratio or constant factor for the geometric sequence of Oostburg for a 5-year time frames?

b. What is the common ratio or constant factor for the geometric sequence of Milwaukee for a 5-year time frames?

c. Is it possible that in the future Oostburg might be larger than Milwaukee? Explain your answer.

d. Why is it unlikely that the population changes for Oostburg and Milwaukee will continue to change as they did from 2010 to 2015?

Comments on the Assessment Task:
The assessment task directs students to summarize and interpret the main components of a geometric sequence. First challenge is to derive the growth rate for Oostburg in a 5-year period of time. The change in population was 5%, thus the common ratio or common factor is 1.05. The decline for Milwaukee indicates that the common ratio or constant factor is 1 – 0.04 or 0.96 for a 5-year period. It is important for students to distinguish that the rate of growth is added to 1 when the population grows and subtracted from 1 when the population declines. With Oostburg increasing in count, and Milwaukee decreasing in count, it is possible that their population counts will equal each other provided the common factors do not change over time. However, it is very unlikely for all conditions to remain the same, given that it would take a long time for the population values to reach the point where Oostburg would exceed Milwaukee. Students are expected to interpret when a population increases and when a population decreases, and the unlikely time frame for conditions that change each population to remain constant.

Additional Assessment Ideas:
The lesson concludes with a summary of why linear models or an exponential model would not continue indefinitely. At some point, a country cannot continue to add population counts without affecting its ability to support the population, or a country cannot continue to lose population without going out of existence. As a formative assessment question, ask students to start describing what might happen in the United States that would slow down the counts or even cause a decrease in the population? (For example, economic conditions that result in having a family very expensive.) Also, what might happen in the United States that would accelerate its growth? Record their ideas from small group discussions. Revisit these ideas as the next model (the recursive model) is developed.