Section 6

Investigating Probability

Investigation 6.1
How Likely Is It?

Overview

This investigation introduces students to basic concepts of **probability,** the mathematics of chance events**.** Examples of **random events** and a method for evaluating the chance they will occur are presented. Specifically, students will begin by discussing events in their lives that are **certain**, **likely**, **neither likely nor unlikely**, **unlikely,** and **impossible** to happen. This will enable them to develop a personal or subjective sense of probability—a measure of how likely an event is to occur. They will then be introduced to the idea of assigning a number from 0 to 1 to the terms certain, likely, neither likely nor unlikely, unlikely, and impossible. Students will use a "walk-on probability scale" to gather and record data on the probability of common events in their lives. This investigation is based on an activity in *Exploring Statistics in the Elementary Grades* by Carolyn Bereska, L. Carey Bolster, Cyrilla A. Bolster, and Richard Scheaffer.

GAISE Components

This investigation follows the four components of statistical problem solving put forth in the *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report.* The four components are formulate a statistical question that can be answered with data, design and implement a plan to collect appropriate data, analyze the collected data by graphical and numerical methods, and interpret the results of the analysis in the context of the original question. This is a GAISE Level A activity.

Learning Goals

Students will be able to do the following after completing this investigation:

- Use the terms certain, likely, unlikely, and impossible correctly
- Associate the chances of occurrence of an event with a position on a probability scale

Common Core State Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them
2. Reason abstractly or quantitatively

3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics

Common Core State Standards Grade Level Content

7.SP.5 Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around 1/2 indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event.

NCTM Principles and Standards for School Mathematics

Data Analysis and Probability

Pre-K–2 All students should discuss events related to students' experiences as likely or unlikely.

Grades 3–5 All students should describe events as likely or unlikely and discuss the degree of likelihood using such words as certain, equally likely, and impossible; predict the probability of outcomes of simple experiments and test the predictions; understand that the measure of the likelihood of an event can be represented by a number from 0 to 1.

Materials

- Masking tape and marker to create a walk-on probability scale
- Table 6.1.1 Chance Events (available on the CD)
- "What are the Chances?" Worksheet 6.1.2 for each student (available on the CD)

Estimated Time

One day

Instructional Plan

Formulate a Statistical/Probabilistic Question

1. Begin this investigation by asking your students if they know what the word "impossible" means. Ask them for several examples of events that

would be impossible for them to perform, such as run a mile in three minutes, build a tower of blocks 100 feet tall, or eat 500 hamburgers in one sitting. Ask them for several examples of events they are certain they can perform, such as add 2 + 2 correctly or write their name. Ask your students to name several events that are between being impossible and certain for them to do, such as doing a jumping rope five times in a row without missing or tossing a wad of paper in a basket six feet away. Explain that, often times, when playing a game like Candy Land or Chutes and Ladders, it is hard to tell what is going to happen. Many games have a spinner, number cube, or other device that gives the number of spaces to move in a turn. Ask your students if they know what the spinner will land on before they spin. Tell your students that they will be exploring events whose actual outcomes they can't always be absolutely certain will happen. Explain to them that such events are referred to as **chance events**. Lead them to the statistical/probabilistic question, "How likely is it for each chance event in a list to happen: impossible to happen, unlikely to happen, neither unlikely nor likely to happen, likely to happen, or certain to happen?"

2. Display Table 6.1.1 and ask your students to review the list of chance events. They need to classify each as being impossible to occur, unlikely to occur, neither unlikely nor likely to occur, likely to occur, or certain to occur.

Table 6.1.1 Chance Events

Classify each of these chance events as being impossible to occur, unlikely to occur, neither unlikely nor likely to occur, likely to occur, or certain to occur.

a. The class will watch TV in school today.

b. We will all use computers sometime today.

c. We will have lunch today.

d. The class will be in school on Saturday.

e. The class will go to the movies this week.

f. We will go outside for recess today.

g. If the teacher were to put the names of all the students in our class in a hat and draw one name, a boy's name will be chosen.

h. If I have a bag of 10 blue cubes and one red cube and draw one cube, the red cube will be drawn.

Collect Appropriate Data

1. Place your students into groups of three or four and give them a recording sheet labeled as follows (available on the CD):

Impossible	Unlikely	Neither Unlikely nor Likely	Likely	Certain

Have each group decide how likely each of the eight chance events in Table 6.1.1 will occur by placing each event under impossible, unlikely, neither unlikely nor likely, likely, or certain. An example is shown in Figure 6.1.1. Note that it might be easier for some students if the chance events are printed on pieces of paper, one per piece, so they can simply move them around to the correct column.

Impossible	Unlikely	Neither Unlikely nor Likely	Likely	Certain
Saturday school	Red cube drawn	Draw boy's name	Watch TV	Use computers
Go to movie			Recess outside	Lunch

Figure 6.1.1 Chance events

2. Ask your students to name other chance events that would fit under impossible, unlikely, neither unlikely nor likely, likely, or certain. Have them add their chance events to the list and explain some of their suggestions to the class as a whole.

Analyze the Data

1. Explain to your students that they have given a word classification to how likely certain chance events are to occur. Lead the discussion to the idea of probability as a method for assigning a number to the words impossible, unlikely, neither unlikely nor likely, likely, certain. On the board, draw the probability scale as shown in Figure 6.1.2. Explain that 0 means there is no chance of occurring, or impossible, and 1 means certain to occur. Place the word unlikely halfway between impossible and the middle,

and place the word likely halfway between the middle and certain. Explain that ½ is halfway between the 0 and 1 and would represent "neither unlikely nor likely."

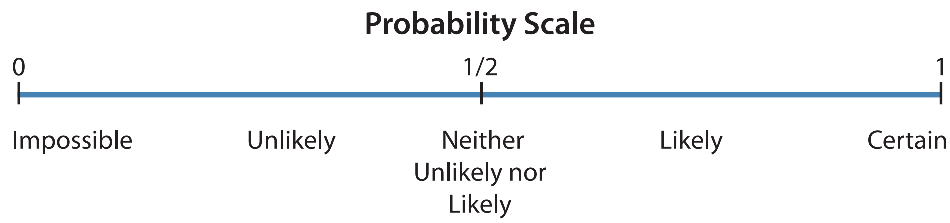

Figure 6.1.2 Probability scale on a number line

2. Place your students in groups of four and give each group a probability scale handout. Ask the students to write the events in their list (the original ones plus the ones they added) on the probability scale. Figure 6.1.3 shows how one class placed the chance events on the probability scale.

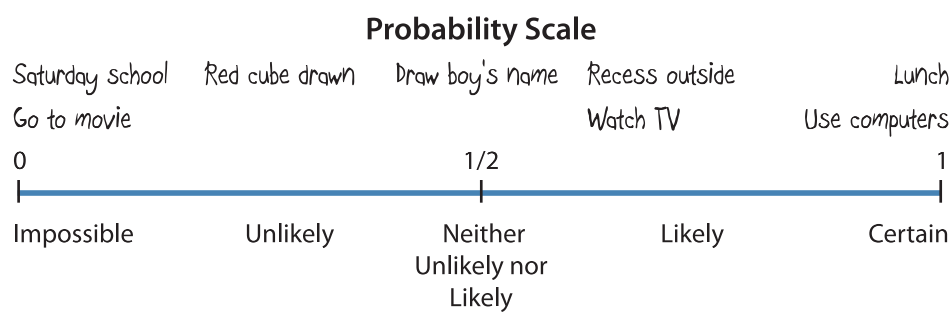

Figure 6.1.3 Sample class results on the probability scale

3. Place a long piece of tape on the floor to make a "walk-on probability scale." Label the piece of tape as shown in Figure 6.1.4.

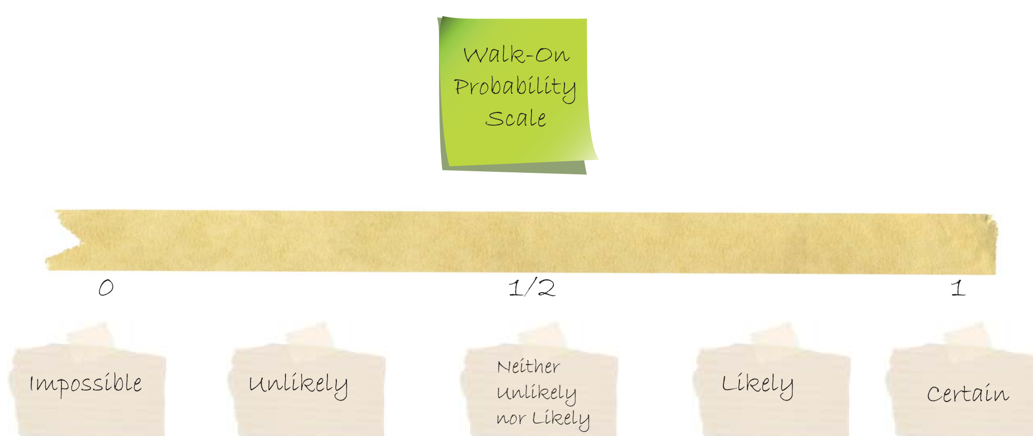

Figure 6.1.4 Walk-on probability scale

For each event or another event that could give different results (e.g., How likely is it that you will be able to jump a jump rope 20 times in a row?), have each of your students stand on the place in the scale that corresponds to their idea about the chance of that particular event occurring.

For many of the events, students will stand in different places on the scale. Have your students identify which events prompted all students to stand in the same place and which prompted students to stand in difference places. Discuss possible reasons for each standing arrangement.

Note that if there is not much variation in their responses, use other chance events such as "How likely is it that your favorite sports team will win their next game?" or "How likely is it that girls are better than boys in shooting free throws?"

Interpret the Results in the Context of the Original Question

Have your students recall that the original statistical/probabilistic question was, "How likely is it for each chance event in a list to happen: impossible to happen, unlikely to happen, neither unlikely nor likely to happen, likely to happen, or certain to happen?" At the start of this investigation, your students shared events they said they couldn't do, some that they were certain they could do, and some events that they were uncertain they could do. Ask your students to summarize what it means for a chance event to be impossible, unlikely, neither unlikely nor likely, likely, or certain to happen. Have them explain their answers using some of the events discussed in this investigation.

Example of 'Interpret the Results'

Note: The following is not an example of actual student work, but an example of all the parts that should be included in student work.

Our teacher asked us to give him examples of things that are impossible for us to do. Impossible means that the event cannot happen. Some of the events we said were hit a baseball 500 feet in the air, fly like a bird, and run 100 miles an hour. He also asked us for events we are certain will happen. These are events that have to happen. Some of us said the sun will rise in the east, I will sit at my school desk today, and I will eat lunch in the lunchroom today. We also talked about events that we were not certain of happening. We said it was likely that we would eat a dessert today and drink milk today, but we said that eating something blue was unlikely because we didn't think it would happen, but it might if someone ate a blue sucker.

After we assigned a word to each of the chance events as to how often we thought they would happen, our teacher had us assign numbers to the events. These numbers are called probabilities. A probability for a chance event is how likely the event will occur. The probability numbers go from 0, which means impossible, to 1, which means certain. So, we assigned 0 to the event that we could run 100 miles an hour, and we assigned 1 to the event the sun will rise in the east. We didn't assign a number to the events we thought were unlikely, but we suggested that they would be between 0 and the middle. The likely ones would be between the middle and 1. Our teacher told us that the halfway number between 0 and 1 is the fraction ½, and it would mean neither unlikely nor likely.

Our teacher put tape on the floor that showed 0, ½, and 1 spread out. All of us in the class stood on the "walk-on probability scale" to show how likely we thought the event "I can jump a jump rope 20 times in a row" would be. It was neat to see that some of us didn't think we could do it. They were down toward 0 and others were spread out between 0 and 1. I was pretty sure I could do it, so I stood about halfway between ½ and 1 on the likely part.

Assessment with Answers

1. Think about each of the following events. Decide where each event would be located on the scale below. Place the letter for each event below on the appropriate place on the scale.

   What are the chances for each event?

   A. The next roll of a fair number cube will be a 2.

   B. You will be successful in four of your next 10 free throw shots.

   C. You will meet a dinosaur on your way home from school.

   D. You will read at least three books this month.

   E. A coin will come up heads five times in a row.

   F. A word chosen randomly from this sentence has four letters.

   G. It will be sunny tomorrow.

   H. You will eat something the color blue today.

   I. A spinner with 10 equal parts numbered 1 through 10 will come up an even number in the next spin.

   J. You will have math homework tonight.

   K. If the names of all the teachers at our school are in a hat, my teacher's name will be picked.

Probability Scale

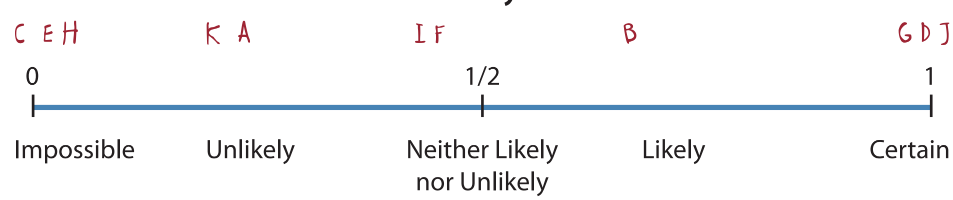

Figure 6.1.5 Sample class results on the probability scale

2. Write two events that are impossible to occur, two that are unlikely to occur, two that are neither unlikely nor likely to occur, two that are likely to occur, and two that are certain to occur. Give reasons for your answers.

   Some possible responses:

   Impossible

   The president of the USA will visit me at home tomorrow.

   I can fly like a bird.

   Unlikely

   The card chosen from a shuffled deck of cards will be a face card.

   There will be a fire drill in school today.

   Neither Likely nor Unlikely

   The number rolled on a fair number cube is even.

   The color of a card chosen at random from a deck of cards is red.

   Likely

   The result of one spin of a spinner with equal colors red, blue, green, and yellow will be a primary color.

   I will play a video game sometime this week.

   Certain

   The sun will rise in the east.

   I will go to bed tonight.

Extension

1. Bring in a board game like Chutes and Ladders or Candyland. Investigate the different outcomes that can occur when playing the game. Ask students to place the outcomes on the probability scale and assign a value of chance for that event to occur.

2. For intermediate students, discuss ways to express probability as percents, fractions, and ratios. What values would they associate with the terms impossible, unlikely, neither unlikely nor likely, likely, and certain? Explain why.

3. Have students use fractions or percents found in newspapers or magazines to make their own scale of likelihood of events.

References

Bereska, C., L. C. Bolster, C. A. Bolster, and R. Scheaffer. 1998. *Exploring statistics in the elementary grades: Book one, grades k–6*. White Plains, NY: Dale Seymour.

Franklin, C., G. Kader, D. Mewborn, J. Moreno, R. Peck, M. Perry, and R. Scheaffer. 2007. *Guidelines for assessment and instruction in statistics education (GAISE) report: A pre-k–12 curriculum framework*. Alexandria, VA: American Statistical Association. *www.amstat.org/education/gaise*.

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Common Core State Standards for Mathematics. www.corestandards.org.

Investigation 6.2
What's the Chance of Seeing an Elephant at the Zoo?

Overview

This investigation focuses on introducing students to answering **"how likely"** questions that are based on data collected and presented in a tally chart. This activity combines basic concepts in data collection and **probability** in a science context, identifying and classifying zoo animals in various ways. Students will use a package of zoo animal crackers to sort them according to some criterion, construct a tally chart, graph data in a bar graph, and use the data collected to determine probabilities.

GAISE Components

This investigation follows the four components of statistical problem solving put forth in the *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report.* The four components are formulate a statistical question that can be answered with data, design and implement a plan to collect appropriate data, analyze the collected data by graphical and numerical methods, and interpret the results of the analysis in the context of the original question. This is a GAISE Level A activity.

Learning Goals

Students will be able to do the following after completing this investigation:

- Demonstrate an ability to sort items, specifically animals, according to likeness
- Display their data in a tally chart/frequency table and illustrate the distribution in a bar graph
- Become familiar with the process of randomly choosing an item from a data set
- Display their result as a fraction of some specific outcome to occur
- Understand basic "how likely" probability terms: least likely, equally likely, most likely Compare probabilities and express one as being more likely or less likely than another to occur
- Recognize and identify categorical data

Science

Students will learn science terms such as claws, hooves, carnivores, herbivores, and omnivores.

Common Core State Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.

Common Core State Standards Grade Level Content

6.SP.1 Recognize a statistical question as one that anticipates variability in the data related to the question and accounts for it in the answers.

6.SP.2 Understand that a set of data collected to answer a statistical question has a distribution that can be described by its center, spread, and overall shape.

7.SP.5 Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around ½ indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event.

7.SP.8a Understand that, just as with simple events, the probability of a compound event is the fraction of outcomes in the sample space for which the compound event occurs.

NCTM Principles and Standards for School Mathematics

Data Analysis and Probability

Pre-K–2 All students should discuss events related to students' experiences as likely or unlikely.

Grades 3–5 All students should describe events as likely or unlikely and discuss the degree of likelihood using such words as *certain, equally likely*, and *impossible*; understand that the measure of the likelihood of an event can be represented by a number from 0 to 1.

Materials

- One package of Austin Zoo Animal Crackers, a Kellogg product, for each pair of students. Each 2 oz. package contains around 30 crackers. **Note:** Other brands may be used, but the types and quantities of animals may differ from the Austin brand used in this investigation.
- Graph paper (large squares)

Estimated Time

One day

Instructional Plan

Formulate a Statistical/Probabilistic Question

1. Discuss with your students animals they might see in a zoo. Tell them that rather than taking a trip to the zoo, the zoo is coming to them. Give each pair of students a bag of zoo animal crackers. Have them open them and identify the animals represented. Make certain the children do not eat their crackers before completing the activity!

   In the Austin Zoo Animal Crackers, there are 12 possible 'zoo' animals: bear, camel, elephant, lion, monkey, mountain goat, owl, penguin, rabbit, rhinoceros, tortoise, and zebra. Figure 6.2.1 shows a picture of the 12 possible animals. Note that each bag may not necessarily contain all 12 species. Also, your students may want to give them other names. For example, the zebra may be labeled a horse. Moreover, they may question whether a rabbit is a zoo animal. Regardless, the class should agree on an assignment of names.

Figure 6.2.1 Twelve animals in Austin Zoo Animal Crackers

2. Ask your students what questions they would like to investigate about their zoo. Some questions students might suggest are the following:

   - How many of each type of animal are there in the zoo?
   - What is the class's favorite animal?

   Through a discussion, lead your students to the statistical/probabilistic question, "If an animal were to be chosen at random from a bag of zoo animals, which type of animal would be the most likely or least likely to be chosen?" Note that the randomness requires that each cracker have the same chance of being chosen as does any other cracker.

Collect Appropriate Data

Ask your students to organize their animals (crackers) by stacking animals of the same species on top of each other. The purpose will be to get a count of how many of each animal they have. One way that students may want to organize their animals is a pictograph, in which animals are lined up horizontally and then those of the same species are placed flat on the desk vertically above each other.

Note that animals are often broken in transit. Discuss with your students that it is important they all agree on what to do with the broken crackers. You might suggest the "broken animals" that can be put back together completely might be counted, whereas those that are broken beyond complete recognition are not to be counted. (Those data and only those data at this point in the investigation may be consumed.)

Analyze Data

1. Have each group of students create a tally chart/frequency distribution (available on the CD) for their zoo data set based on a single bag of crackers. **Note:** A frequency distribution is a list of the values (categories or names) of a categorical variable and their frequencies (how many times each occurs). In this example, the categories are animals. Table 6.2.1 shows an example of a frequency distribution constructed by one group of students.

Table 6.2.1 Example of a Frequency Distribution of a 'Zoo'

Name of Animal	Tally	Count/Frequency
Bear	|	1
Camel	||	2
Elephant	||||	4
Lion	|	1
Monkey	𝍸|	6
Mountain Goat	||	2
Owl	||	2
Penguin	|||	3
Rabbit		0
Rhinoceros	||	2
Tortoise	|||	3
Zebra	||||	4

2. Using the tally chart/frequency table, have your students draw a bar graph using graph paper. Be sure there is an equal amount of space separating the names of the animals listed below the horizontal axis. The vertical scale should be labeled by frequency of animals. Figure 6.2.2 is a bar graph for the example zoo collected by one group of students.

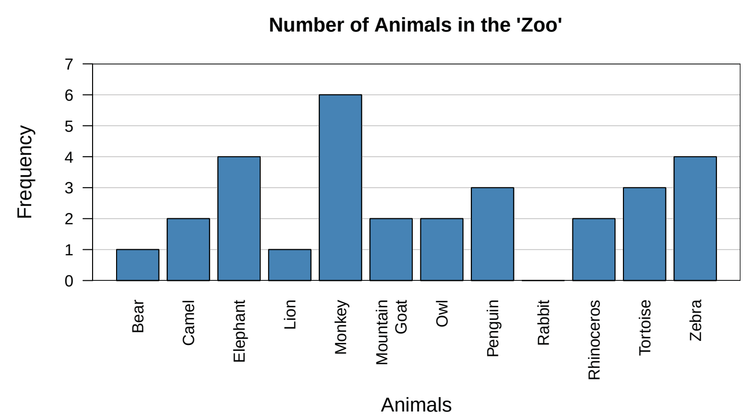


Figure 6.2.2 Bar graph of the example zoo data

Note: Students could make a picture graph. Using large square graph paper, students label each animal type on the horizontal axis like the bar graph example shown. The students then place one animal in each square above the appropriate label. If there were three penguins, then each penguin would be placed in a square above the "Penguin" label. Using the graph paper ensures that the height of each column is the same as the frequency of each animal type so that three penguins occupy the same area space as do three tortoises, for example.

3. Ask your students the following questions:

   a. How many elephants were in their zoo?

   b. Which animal has the most of its kind in the zoo? That animal is called the mode. Note that the monkey is the mode animal in the example.

   c. Which animal has the least of its kind in the zoo?

4. Tell your students to imagine they were to reach into their bag of animals and, without looking, mix up all the animals and then pick out one. Ask your students what animal is the most likely to be drawn out. Discuss how the "most likely" is the animal with the greatest number and the "least likely" is the animal with the smallest number. Note that some students may say the least likely is the rabbit, since there aren't any. Others may argue that the smallest number has to be positive, so the bear or lion would be the smallest number in the example.

5. Combine the class data on the board in a frequency distribution. Make a bar graph of the class data. Figure 6.2.3 shows the results from the whole class.

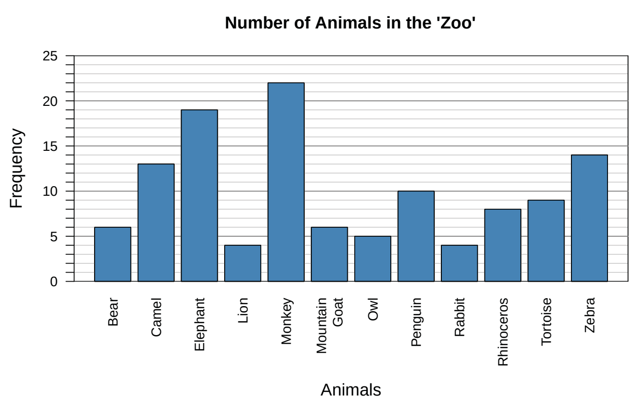


Figure 6.2.3 Bar graph of sample class zoo

6. Ask your students to identify the most likely and least likely animal to be selected at random from a container that would contain all the animals collected in the class. Have them explain their choices.

7. Ask your students to compare the bar graph representing their bag or zoo to the bar graph of the whole class's results.

8. Ask them which animal was most likely to be drawn from their bag and compare their answer to the class's result.

Interpret the Results in the Context of the Original Question

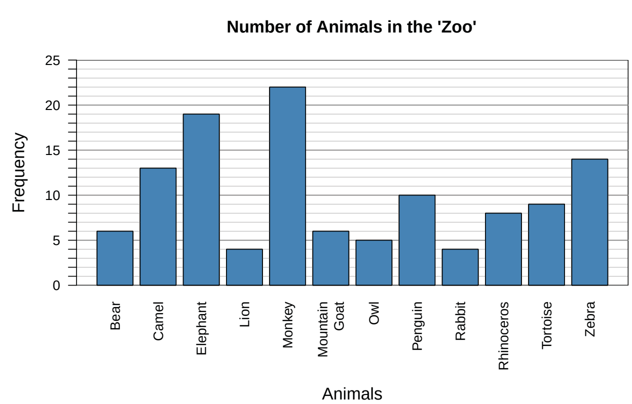
1. Have your students recall the original statistical/probabilistic question, "If an animal were to be chosen at random from a bag of zoo animals, which species would be the most likely or least likely to be chosen?" Ask your students to write an answer to the question and use the class data to justify their answer.

2. Ask students if they think Kellogg makes the same number of each animal in their production of Austin Zoo Animal Crackers.

Example of 'Interpret the Results'

Note: The following is not an example of actual student work, but an example of all the parts that should be included in student work.

Our teacher talked to us about animals in a zoo. We made a list and then our teacher handed out a bag of animal crackers that represented our zoo. We worked in pairs. We investigated the question, "If an animal were to be chosen at random from a bag of zoo animals, which type of animal would be the most likely or least likely to be chosen?" We made a tally chart of the animals in our bag and then made a bar graph. Based on our bar graphs, we decided which animal was most likely by looking at the heights of the bars. The animals that had the highest bars were the ones we thought would be the most likely to be chosen. Also, the animal that occurred the most is called the mode. We combined the results from all the groups. The following bar graph shows our class results:



Our class data showed that the monkey was the most likely to be chosen, with the elephant also a good possibility. So the monkey is the mode animal, while the lion, owl, and rabbit were the least likely animals to be chosen. Also, we concluded that Kellogg's does not bake the same number of each animal. If they did, the bars would be more even. Statistics is kind of fun because we were allowed to eat our data when we were done.

Assessment with Answers

Chris sorted a bag of animal crackers and drew the bar graph shown in figure 6.2.4.

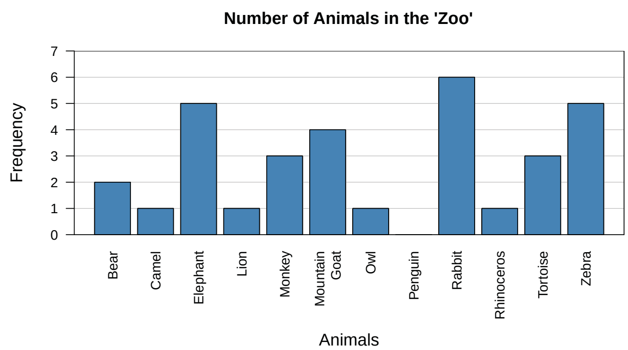


Figure 6.2.4 Bar graph of Chris's zoo

1. How many elephants were in Chris's bag? There were five elephants in Chris's bag.

2. How many rabbits were in Chris's bag? There were six rabbits in Chris's bag.

3. How many more rabbits were in the bag than penguins? There were six rabbits and no penguins, so there were six more rabbits than penguins in Chris's bag.

4. How many more rabbits were in the bag than monkeys? There were three more rabbits than monkeys in Chris's bag.

5. If you reach into Chris's bag and randomly picked out one animal, which animal would:

   Most likely be chosen? The rabbit occurred the most and hence would be the most likely animal chosen.

   Least likely be chosen? Well, if the answer has to be positive, then it is a tie among camel, lion, owl, and rhinoceros. If 0 is allowed, then the least likely would be penguin, since there are no penguins.

Extensions

1. Using the class data, ask your students to calculate the probability of randomly choosing each animal type.

2. Ask what the probability is of randomly choosing:
    - An animal with four feet?
    - An animal with claws? Hooves?
    - An animal with knees?
    - A meat-eating animal? A pure carnivorous animal?
    - An animal that eats vegetation? A pure herbivorous animal?
    - An animal that eats both meat and vegetation? A pure omnivorous animal?

3. Discuss how to draw a Venn diagram. For example, create a Venn diagram identifying the carnivores, herbivores, and omnivores for a package of animal crackers as a class. Then, in groups of two, have students create their own Venn diagram. Figure 6.2.5 is based on different traits chosen by a pair of students.

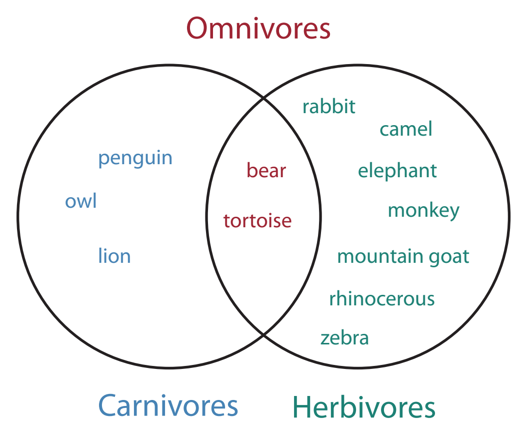


Figure 6.2.5 Venn diagram of carnivores, herbivores, and omnivores

References

Franklin, C., G. Kader, D. Mewborn, J. Moreno, R. Peck, M. Perry, and R. Scheaffer. 2007. *Guidelines for assessment and instruction in statistics education (GAISE) report: A pre-k–12 curriculum framework*. Alexandria, VA: American Statistical Association. *www.amstat.org/education/gaise*.

National Council of Teachers of Mathematics. 2000. *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Common Core State Standards for Mathematics. www.corestandards.org.

Investigation 6.3
What Do Frogs Eat?

Overview

This investigation introduces students to answering questions about proportionality that leads to a basic understanding of probability in a science context (food choices for frogs). Students make conjectures about the results of experiments and test them with simulations. They compute **experimental probabilities** for **simple events** using tally charts and bar graphs. Students use a container (e.g., paper bag) with three small objects (e.g., colored cubes) to sort outcomes from a number of draws from their bag according to some criterion (food choices). Then, they construct a tally chart, graph data in a bar graph, and use the data collected to determine probabilities.

GAISE Components

This investigation follows the four components of statistical problem solving put forth in the *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report.* The four components are formulate a statistical question that can be answered with data, design and implement a plan to collect appropriate data, analyze the collected data by graphical and numerical methods, and interpret the results of the analysis in the context of the original question. This is a GAISE Level B activity.

Learning Goals

Students will be able to do the following after completing this investigation:

- Make conjectures about probabilities of simple events according to some criterion
- Organize categorical data in a tally chart and graph an empirical distribution using a bar graph
- Display their result as a fraction of some specific outcome to occur and test their conjectures
- Compare fractions by comparing empirical probabilities (expressed as fractions) and associate larger (smaller) fractions with events being more likely (less likely) to occur
- Learn about small frogs' living environment and eating habits (mostly insects, worms, and snails)

Common Core State Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.

Common Core State Standards Grade Level Content

7.SP.5 Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around ½ indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event.

7.SP.7b Develop a probability model (which may not be uniform) by observing frequencies in data generated from a chance process.

7.SP.8 Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation.

NCTM Principles and Standards for School Mathematics

Data Analysis and Probability

Grades 6–8 All students should formulate questions, design studies, and collect data about a characteristic shared by two populations or different characteristics within one population; use observations about differences between two or more samples to make conjectures about the populations from which the samples were taken; understand and use appropriate terminology to describe complementary and mutually exclusive events; use proportionality and a basic understanding of probability to make and test conjectures about the results of experiments and simulations; compute probabilities for simple compound events using such methods as organized lists, tree diagrams, and area models.

Number and Operations

Grades 6–8 All students should work flexibly with fractions, decimals, and percents to solve problems; understand and use ratios and proportions to represent quantitative relationships; select appropriate methods and tools for computing with fractions and decimals from among mental computation, estimation, calculators or computers, and paper and pencil, depending on the situation, and apply the selected methods.

Materials

- Small blue, green, and yellow cubes
- Small brown paper bags (three for each group of four students)
- ½-inch square graph paper sheet for each student
- Recording form (three for each group) (available on the CD)

Estimated Time

One day

Instructional Plan

Formulate a Statistical/Probabilistic Question

1. Discuss with your students the different natural habitats of frogs and their favorite food choices.

2. Explain that, in some habitats, there are more worms than insects or more spiders than snails. So the frogs' diets will depend on what is easily available. Form groups of students, four to a group. Tell them that each group will be getting three bags representing three types of habitat: bag 1, marsh; bag 2, stream; bag 3, tropical garden. In each bag, there are colored cubes representing type of food available in that habitat: blue, flies; green, worms; yellow, snails. The proportions of types of food may be different, appropriately determined by the habitat.

3. Ask your students to formulate an appropriate statistical/probabilistic question based on the background you have given them so far. Lead them to the question, "Is the probability of choosing blue, green, yellow cubes different from bag to bag?" Or, in terms of the frog scenario, "Does habitat have any influence on a frog's food choice?"

Collect Appropriate Data

1. Place your students into groups of four. Give each group three bags labeled 1, 2, and 3. Do NOT show them the contents of the bags, although you should write on the board that blue cubes represent flies, green cubes

Frogs range from the tropics to subarctic regions of the world. There are more than 5,000 species. Adult frogs are carnivores. This means they eat mostly insects like flies, worms, spiders, small fish, and snails. Frogs have no teeth, so they swallow their entire meal because they can't chew. **Note:** Websites about frogs are listed in the references for this investigation.

represent worms, and yellow cubes represent snails. Students are NOT to look inside the bags. **Note:** For YOUR eyes only, the composition of the bags are:

- Bag 1 (marsh): 2 blue, 1 green, and 1 yellow
- Bag 2 (stream): 1 blue, 1 green, and 1 yellow
- Bag 3 (tropical garden): 1 blue, 2 green, and 1 yellow

2. Explain to your students that they will be choosing cubes from the bags in order to determine the proportion of flies, worms, and snails each habitat has. Tell them they must choose their cubes randomly. Remind them that **randomly selected** means each cube in the bag has the same chance of being chosen. Demonstrate for them that to guarantee randomness, the bag needs to be shaken well and that the cubes need to be exactly the same size and shape.

3. Starting with Bag 1 (marsh), your students should randomly draw a cube from the bag and record the result on the recording form. **Note:** Each group should have a recording form for each bag. Have the students replace the cube, shake the bag well, and repeat for a total of 20 draws from the bag. Such choices in statistics are called <u>trials</u>. The students should repeat this procedure for Bags 2 (stream) and 3 (tropical garden), remembering to record their trial results on the other recording forms as shown in Table 6.3.1. (The complete recording form is available on the CD.)

Table 6.3.1 Example of a Recording Form (Partial)

Bag Number ________	
Trial Number	**Results**
1	
2	
3	
4	
…	
20	

✏ Analyze the Data

1. Ask your students to tally their trial results on the recording sheet as shown in Table 6.3.2.

Table 6.3.2 Example of a Frequency Table of Experimental Trial Results Drawn from Bag 1

Bag Number 1		
Color	**Frequency**	**Relative Frequency**
Blue	8	
Green	5	
Yellow	7	
TOTAL	**20**	

2. Ask your students to complete the relative frequency column in each table by dividing the color count by 20. For example, the relative frequency of blue is 8/20.

3. Have your students create three bar graphs showing their trial results separately for each of the three bags. Have your students use graph paper to construct the bar graphs. Note: Make sure your students use the correct procedure to construct a bar graph (i.e., an equal amount of space separating the colors (categories) on the horizontal axis; the vertical scale marked 0 to 1 for relative frequency). An example of a bar graph for the experimental results from Bag 1 is shown in Figure 6.3.1.

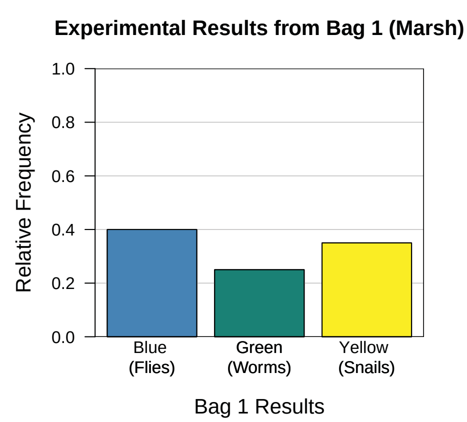


Figure 6.3.1 Bar graph of example experimental results drawn from Bag 1 (Marsh)

4. Explain to your students that each fraction is the **experimental probability** of selecting a specific color (food type) at random from a specific bag (habitat).

5. Ask your students the following questions using their bar graphs and results tables.

   - What is the most likely food choice from each habitat?

- What is the experimental probability of selecting the most likely food choice from each habitat?
- What is the experimental probability of selecting a fly from each habitat?
- What is the experimental probability of selecting a worm from each habitat?
- What is the experimental probability of selecting a snail from each habitat?

6. Explain to your students that having more data than what 20 trials yields gives better estimates of the experimental probabilities of the food types per habitat. So, combine the class data for each bag on the board in a tally chart and then in a bar graph for each bag. Figure 6.3.2 shows a sample of class data for Bag 1.

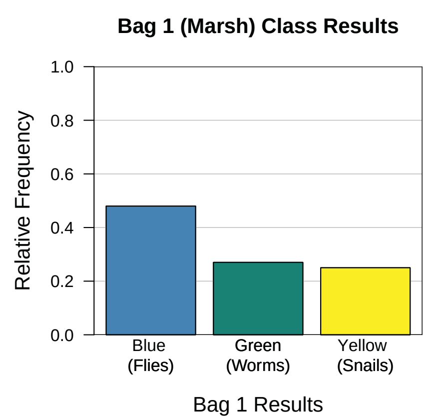


Figure 6.3.2 Bar graph of sample class data for Bag 1 (Marsh)

7. Using the bar graphs, ask your students what they think the distribution of food types in each of the habitats should be. Based on the experimental results, what are estimates of the population proportions of flies, worms, and snails in each of the marsh, stream, and tropical garden habitats?

Interpret the Results in the Context of the Original Question

Have your students write a report that provides an answer to the original question, "Is the probability of choosing blue, green, yellow cubes different from bag to bag?" Or, in terms of the frog scenario, "Does habitat have any influence on a frog's food choice?" Their answer needs to be supported by experimental evidence including trial data and graphs.

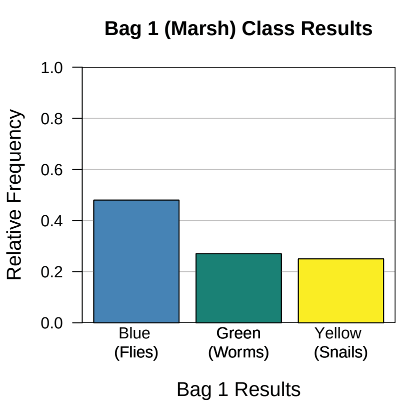
Example of 'Interpret the Results'

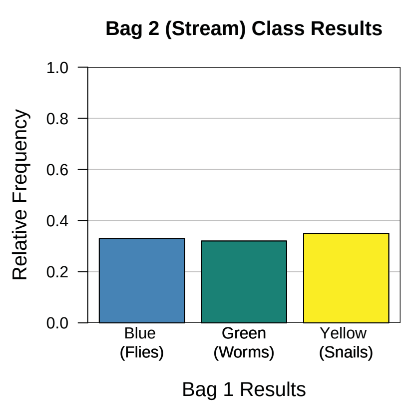
Note: The following is not an example of actual student work, but an example of all the parts that should be included in student work.

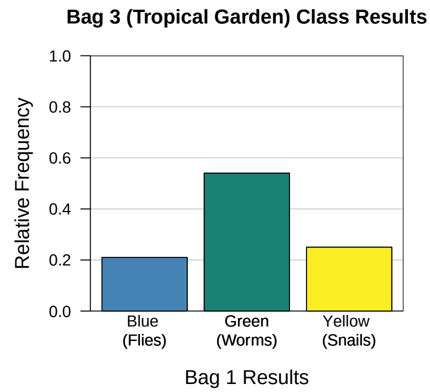
Our biology and mathematics teachers must have gotten together and decided to give us a statistical problem involving trying to determine the distribution of food types for different habitats in which frogs live. The food types that frogs eat are flies, worms, and snails and the frog habitats we used were marshes, streams, and tropical gardens.

To collect data, flies were represented by blue cubes, worms by green ones, and snails by yellow. The habitats were paper bags labeled 1, 2, and 3. The question we investigated was, "Is the probability of choosing blue, green, yellow cubes different from bag to bag?" Or, in terms of the frog scenario, "Does habitat have any influence on a frog's food choice?"

We worked in groups of four and each group was given three bags. The bags had different proportions of colored cubes in them. We were not allowed to look inside the bags. To determine experimental probabilities for the food types per bag, each of our groups randomly selected a food type from a bag, wrote it down, replaced the cube, and did it 19 more times. Each time was called a trial. We shook the bag a lot each time so we didn't bias the choices. To get a better idea of what each bag (habitat population) had in it, we put all of our group results together and drew these bar graphs, one for each habitat.







We decided that Blue (flies) was the most likely pick from Bag 1 (marsh habitat) and that the other two foods, worms and snails, were about the same. So we thought the distribution in marshes would be 45% flies, 27.5% worms, and 27.5% snails.

It turned out that Bag 1 contained 2 blue, 1, green, and 1 yellow, so the population proportions were actually 50%, 25%, 25%. Our experimental results were pretty close.

In the stream habitat, our experimental results indicated that all three food types were equal. We were right because Bag 2, the stream habitat, contained one of each of the colors, so the distribution of food types is 33 1/3% each.

For Bag 3, the tropical garden habitat, our bar graph of experimental results looked like it would have been determined from a 1-2-1 distribution (i.e., 25% flies, 50% worms, and 25% snails). We were right on that one, too. By the way, our individual group results were not really close to the actual bag proportions, but we learned that getting a larger data collection by putting all our group results together brought us much closer to the right answers. It was neat to see that biology and statistics go together.

Assessment with Answers

One group of students drew a cube from each of three bags that were labeled Bag 1 Marsh, Bag 2 Stream, Bag 3 Tropical Garden. They repeated the drawing 20 times for each bag. They got the results shown in Table 6.3.3.

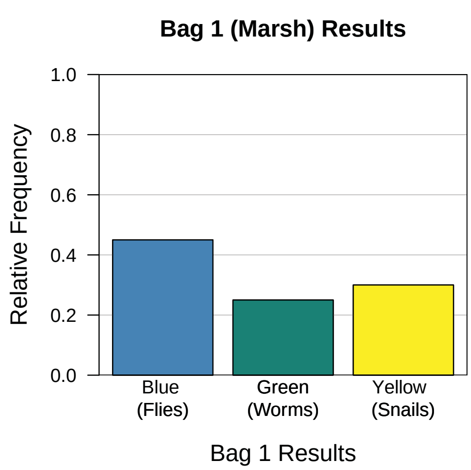
Table 6.3.3 Results of 20 Draws from 3 Bags

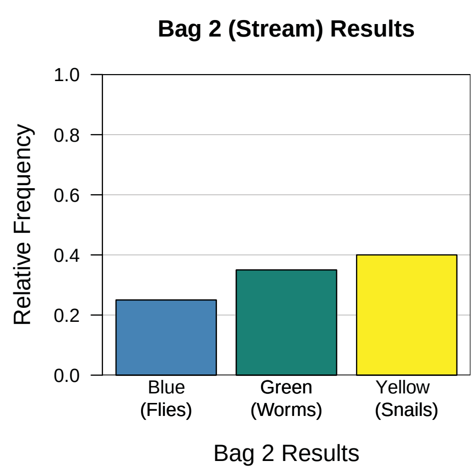
Color	Count for Bag 1 (Marsh)	Color	Count for Bag 2 (Stream)	Color	Count for Bag 3 (Tropical Garden)
Blue	9	Blue	5	Blue	5
Green	5	Green	7	Green	9
Yellow	6	Yellow	8	Yellow	6
TOTAL	**20**	**TOTAL**	**20**	**TOTAL**	**20**

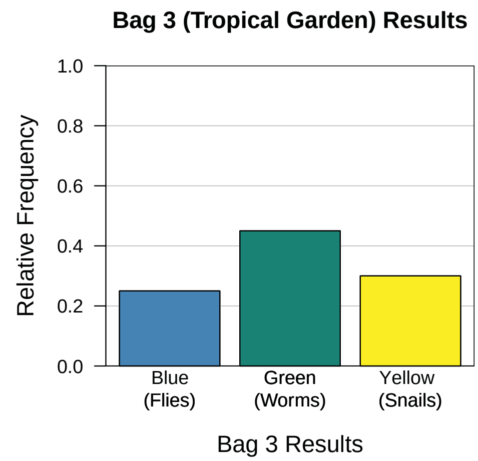
1. Complete the following table by converting each color count into relative frequency.

Color	Count for Bag 1	Rel. Freq.	Color	Count for Bag 2	Rel. Freq.	Color	Count for Bag 3	Rel. Freq.
Blue	9	.45	Blue	5	.25	Blue	5	.25
Green	5	.25	Green	7	.35	Green	9	.45
Yellow	6	.30	Yellow	8	.40	Yellow	6	.30
TOTAL	**20**		**TOTAL**	**20**		**TOTAL**	**20**	

2. Draw a bar graph of the results for each of the three bags. Be sure to include a title for the table and label the axes.







3. If you reach into the student's first bag and randomly choose a single cube, which color would you most likely choose? Explain your answer.
   Blue. Blue has the highest relative frequency. It is the mode color.

4. If you reach into the student's second bag and randomly choose a single cube, which color would you most likely choose? Explain your answer.
   Yellow (.4), but it might be green (.35).

5. If you reach into the student's third bag and randomly choose a single cube, which color would you most likely choose? Explain your answer.
   Green. Green has the highest relative frequency. It is the mode color.
6. Based on these sample results, what do you think the proportion of blue, green, and yellow cubes in each of the three bags is? Explain your answer.
   It looks like the first bag might be 50-25-25%; the second bag might be uniform (evenly distributed); and the third bag might be 25-50-25%, but I would like to have more data to be sure.

Extension

Note: These questions refer to the data in the assessment.

1. If you reach into the student's second bag and randomly choose a single ball, what are the chances of getting a blue or green ball? Explain your answer. 12/20. Add the number of blue and number of green and divide by total number, 20.

2. If you reach into the student's third bag and randomly choose a single ball, what are the chances you do not get a blue ball? Explain your answer.
   15/20. Add the number for the other two colors and divide by 20.

3. Which bag would you choose to draw one cube from to give you the best chance of drawing a green cube? Explain your answer. Bag 3. The number of green is higher than the other two colors.

Additional Extensions

1. Have each group of students construct their own "habitats" (bags with small cubes). Each group should then write probability questions based on their habitats. Allow groups to change the type of animal(s) eating and being eaten. The groups can then exchange their bags and questions.

2. Play the Leap Frog Game at *www.beaconlearningcenter.com/WebLessons/LeapFrog/default.htm*. The object of the game is to "leap" the frogs from one side of the pond to the other. The color of frog resulting from a spin of a spinner gets to "leap" another step across the pond. The first frog to get to the other side wins the game. Which spinner should be chosen to win the game?

3. Give students a "secret" habitat whose contents are not revealed. Students are to repeat the experiment, collect data, and then make a prediction about the number of snails, worms, and flies, knowing only the total number of cubes in the bag. Students should justify their decisions with numbers and sentences.

References

Franklin, C., G. Kader, D. Mewborn, J. Moreno, R. Peck, M. Perry, and R. Scheaffer. 2007. *Guidelines for assessment and instruction in statistics education (GAISE) report: A pre-k–12 curriculum framework*. Alexandria, VA: American Statistical Association. *www.amstat.org/education/gaise*.

National Council of Teachers of Mathematics. 2000. *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Common Core State Standards for Mathematics. www.corestandards.org.

Leap Frog Game. *www.beaconlearningcenter.com/WebLessons/LeapFrog/default.htm*.

USGS Frog Quizzes. *www.pwrc.usgs.gov/frogquiz.*

USGS Patuxent Wildlife Research Center. *www.pwrc.usgs.gov.*

Investigation 6.4
How Many Spins to Win the Prize?

Overview

This investigation focuses on determining, by **experimentation**, the number of spins necessary in a spinner experiment to observe a "success." Students will conduct an experiment and combine class data into tables and graphs, draw conclusions, and make predictions based on the class data collected.

GAISE Components

This investigation follows the four components of statistical problem solving put forth in the *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report.* The four components are formulate a statistical question that can be answered with data, design and implement a plan to collect appropriate data, analyze the collected data by graphical and numerical methods, and interpret the results of the analysis in the context of the original question. This is a GAISE Level B activity.

Learning Goals

Students will be able to do the following after completing this investigation:

- Organize the results of a probability experiment in a frequency table
- Summarize the results of a probability experiment
- Make predictions based on the results of a probability experiment
- Be aware that the more they repeat an experiment, the more reliable their results will be

Common Core State Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.

Common Core State Standards Grade Level Content

7.SP.6 Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability. For example, when rolling a number cube 600 times, predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times.

NCTM Principles and Standards for School Mathematics

Data Analysis and Probability

Grades 6–8 Use proportionality and a basic understanding of probability to make and test conjectures about the results of experiments and simulations

Materials

- Spinner with four equal sections
- Large paper clip
- Recording sheet (five for each group of two students) (available on the CD)
- Grid paper

Estimated Time

1–2 days

Instructional Plan

Formulate a Statistical/Probabilistic Question

1. Begin this investigation by handing out or displaying the "Winning a Silver Car" scenario (available on the CD). Introduce the "Winning a Silver Car" scenario by asking your students if they have ever played a game at a carnival in which they had to spin a spinner to win a prize.

    Winning a Silver Car
    At the school carnival, there is a game in which students spin a large spinner. The spinner has four equal sections: silver, green, blue, and red. Each section represents the color of a toy car that can be won. To play the game, Sarah has to buy some tickets at the ticket booth. She needs one ticket each time she spins the spinner. She also wants to win a silver toy car. If the spinner stops on silver on her first spin, Sarah wins. If not, she has to spin the spinner until it stops on silver. So, she needs to decide how many tickets she should buy to play this game to win a silver toy car.

2. Display the spinner with four equal sections labeled green, silver, blue, and red as shown in Figure 6.4.1. Ask your students to list all the possible

outcomes for each spin and then ask what the chances are of landing on silver with each spin of the spinner.

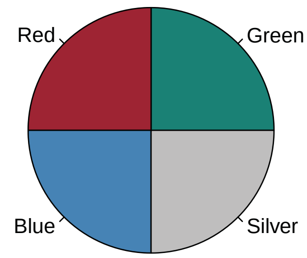


Figure 6.4.1 Spinner

3. Ask your students to formulate a statistical or probabilistic question that will help Sarah. Note that they should formulate the question "How many tickets are needed for Sarah to buy in order for her to win her favorite prize of the silver toy car in playing Winning a Silver Car?"

Collect Appropriate Data

1. Place your students in groups of two. Give each group a copy of the spinner, a paper clip, and a recording sheet (available on the CD). Model one trial for your students. Use a large paper clip as your spinner, spin the spinner until you get silver, and be sure to record the color of each spin on a recording sheet similar to Table 6.4.1. Finally, record the total number of spins needed to get silver.

Table 6.4.1 Recording Sheet

Spin Number	Color
1	
2	
3	
4	
5	
…	

2. Have each group spin their spinner until a silver outcome occurs. One student in the group should do the spinning while the other records the color result of each spin. Remind your students that their spinner should be on a flat surface. Be sure that they record the color outcome of each spin on their recording sheet. Even though the outcome of each spin is being recorded, it is the total number of spins needed to get silver that will be collected from each group. **Note:** Be sure the students are spinning their paper clips correctly. The clip should rotate several times before coming to rest. This may take some practice to avoid biasing the results.

3. Create a number line that begins at 1 and ends at the most number of spins. To determine the most number, ask your students if any group required more than 20 spins, 15, 14, 13, … Then, draw a number line on the board. Ask each group for their number of spins. Represent each group's number of spins with an X above the number of spins to create a dotplot. Figure 6.4.2 is an example of a class dotplot after each group completed the simulation once.

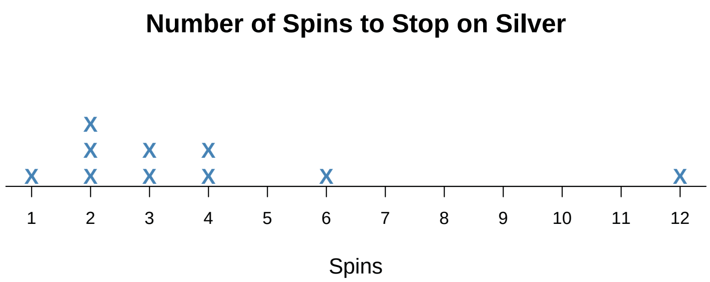


Figure 6.4.2 Dotplot of an example of class data

4. Explain to your students that the more data they have, the better their estimate for Sarah will be. Direct the groups to repeat the simulation again. Have the students reverse who spins and who records. Be sure the new students practice their spinning. As the groups complete the simulation, add their results to the class dotplot. Have students repeat the simulation at least four more times, each time adding the students' results to the class dotplot. An example of a class dotplot based on 100 simulations is shown in Figure 6.4.3.

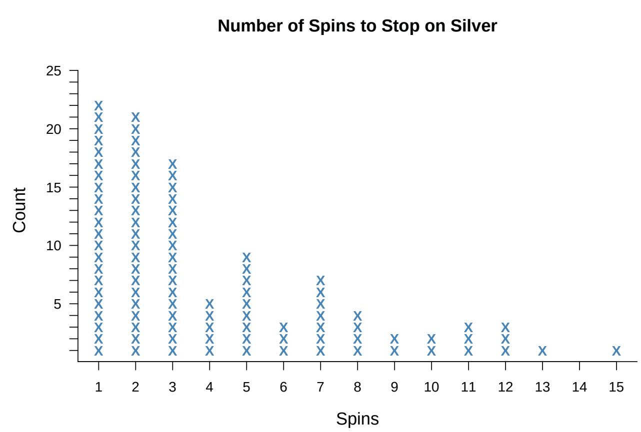


Figure 6.4.3 Dotplot of class data number of spins

Analyze the Data

1. Ask your students to describe the class dotplot. Be sure they understand what the horizontal and vertical axes are measuring. **Note:** The distribution is skewed to the right.

2. Have your students answer the following questions by using the dotplot:

   a. What was the least number of spins to win the silver toy car?

   b. What was the largest number of spins needed to win the silver toy car?

   c. What number of spins happened most often (the mode)?

   d. What is the median number of spins to win the silver toy car?

   e. Estimate the average number of spins needed to win the silver toy car. Explain how you made your estimate. Note that the "fair share" approach to the mean was described in Section 3.3 and the "balance of deviations" development of the mean was discussed in Section 3.4.

3. Ask your students whether they think Sarah having to play the game eight (or a value in the "middle" of the number of spins) times before she won the silver car is a lot of spins. Have them explain their answer.

4. Discuss with your students what their recommendation would be to Sarah for the number of tickets she should buy. For example, based on their simulated distribution (dotplot) what is the probability that Sarah will need 20 (or some value beyond the simulated results) or more tickets? **Note:** This can be written P(20). Since the highest number of tickets simulated was 15 in our class example, the likelihood that she needs 20 tickets is basically zero, or P(20) = 0.

5. What about 15 (or your highest value obtained) or more tickets? The probability that she will need 15 or more tickets, written P(15 or more), is small since that only occurred one time in 100 trials. In other words, if she were to play the game 100 times, she would need 15 or more tickets only once.

6. Discuss with your students how to determine the probability of needing to buy 12 or more tickets, written P(12 or more). Emphasize that 12 or more means the P(12) + P(13) + P(14) + P(15) or $P(x \geq 12) = 3/100 + 1/100 + 0/100 + 1/100 = 5/100 = .05$. In other words, if Sarah were to play the game 100 times, she will probably need at least 12 tickets on five of those times.

7. Ask your students to create a table of probabilities based on their simulated results (available on the CD). The table should have three columns: Number of Spins Needed to Win (n), P(n), and P(n or more). See Table 6.4.2 for the sample class simulations.

Table 6.4.2 Probabilities Based on the 100 Simulations

Number of Spins Needed to Win (n)	P(n)	P(n or more)
15	.01	.01
14	.00	.01
13	.01	.02
12	.03	.05
11	.03	.08
10	.02	.10
9	.02	.12
8	.04	.16
7	.07	.23
6	.03	.26
5	.09	.35
4	.05	.40
3	.17	.57
2	.21	.78
1	.22	1.00

8. After your students have completed the table, display the results on the board. Point to different values under the P(n) column and ask them to explain how they found their answer. For example, in Table 6.4.2, the value .09 in the row for five spins to win was found by taking the number of times a simulation resulted in needing five spins and dividing by 100.

9. Point to different values under the P(n or more) column and ask your students to explain how they determined their answer. For example, in Table 6.4.2, the value .12 in the row for nine spins needed to win was found by adding P(15) + P(14) + P(13) + P(12) + P(11) + P(10) + P(9) = .01 + .00 + .01 + .03 + .03 + .02 + .02 = .12.

10. Ask students to explain why the P(1 or more) = 1?

11. Ask your students to make a recommendation to Sarah as to the number of tickets she should buy. Note that their answers will vary. A possible answer is that she should buy seven tickets because she would need more tickets only about 25% of the time (i.e., she should be safe 75% of the time).

Interpret the Results in the Context of the Original Question

1. Have your students recall the original statistical question, “How many tickets are needed for a player to win his/her favorite prize in playing

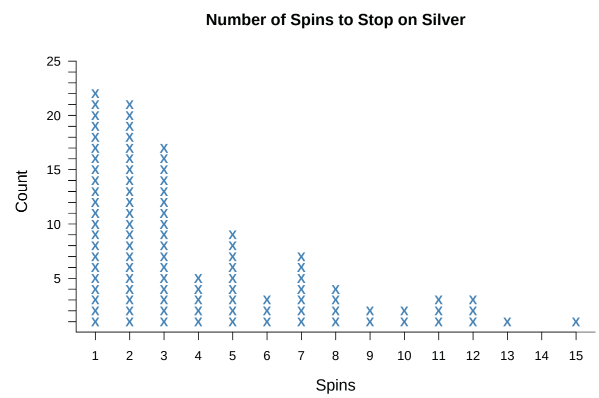
Winning a Silver Car?" Have your students write an answer to the question and justify it using the graphs, the calculations, and their analysis.

2. Ask your students if another class's results would be similar if they spun the spinner a large number of times.

Example of 'Interpret the Results'

Note: The following is not an example of actual student work, but an example of all the parts that should be included in student work.

We were given the Winning a Silver Car scenario, in which a girl named Sarah wanted to win a silver car by spinning a spinner that had four equal sections. One section was silver, and she had to use a ticket for each spin. We wanted to find out how many spins it would take for Sarah to win a silver car. To help answer this question, we played the game by spinning the spinner and recording how many spins it took before the spinner stopped on silver. As we played the game, we recorded on a dotplot how many spins it took to stop on silver. We played the game 100 times, and the results are shown in the following dotplot.



We used our class data and found that the median of our number of spins needed to win was three. We also estimated the mean to be about four. We next calculated the probability of winning for each number of spins and the probabilities for each number of spins or more. Based on the table of probabilities, we thought Sarah would need to buy about seven tickets to win a silver car. About 75% of the time, our dotplot showed Sarah would win if she bought seven or fewer tickets.

Assessment with Answers

A carnival game used a spinner with five equal sections (Figure 6.4.4). A person won a prize if the spinner stopped on the yellow section. One hundred students each played the game until they won a prize. This means they each kept spinning the spinner until it stopped on yellow. Figure 6.4.5 is a dotplot of how many spins it took each of the 100 students to win a prize.

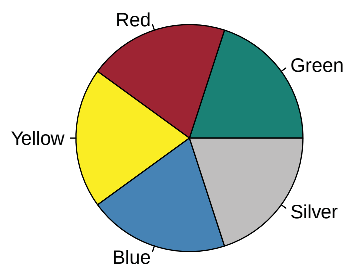


Figure 6.4.4 Spinner

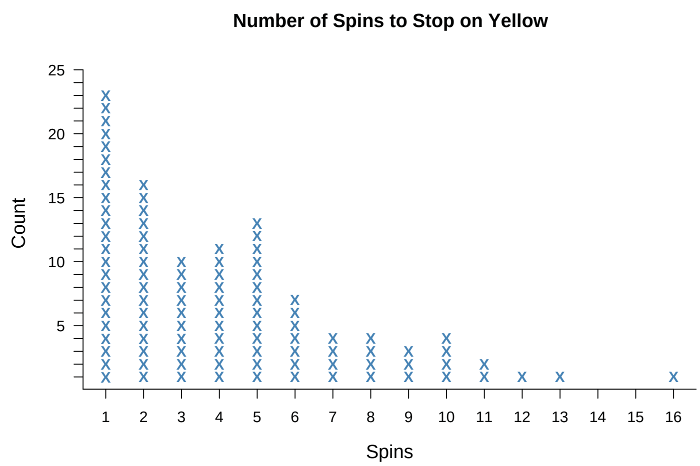


Figure 6.4.5 Dotplot of number of spins needed to stop on yellow

1. Describe the distribution of the number of spins to stop on yellow. The distribution is skewed to the right.

2. Estimate the center of the distribution and explain what this value would represent. The median is four, which is the middle number, and the mean is also approximately four, which is the point at which the distribution would balance. The mode is one, which has the highest number of occurrences.

3. Find the probability of each of the following:
    - P(exactly 8 spins) = 4/100 = .04
    - P(exactly 3 spins) = 10/100 = .10
    - P(11 or more spins) = 5/100 = .05
    - P(1 or more spins) = 1

4. Andrea, a sixth-grade student, played the game. Use words, numbers, and/or drawings to explain how many times you think it would take Andrea to play the game. I think Andrea will take about six spins before the spinner stops on yellow. The median and mean are both about four, and the dotplot shows the probability of the number of spins needed for the spinner to stop on yellow seven or more times to be 20%. So, 80% of the time, the number of spins before the spinner stopped on yellow is six or fewer.

Extensions

1. Increase the number of equal sections on the spin. For example, create a spinner with six equal sections and ask your students if they think it would take about the same number of spins to get a silver car as on the spinner with four equal sections. Have them spin the new spinner a large number of times, collect class data, and construct a dotplot. Have your students compare this plot with the class dotplot from the four equal section spinner.

2. Using the four equal section spinner, ask your students how many spins they think it will take to get all four colors. Have them conduct a simulation of how many spins until they get all four colors. Collect the data and construct a class dotplot.

3. Have your students go to *www.mathwire.com/data/CerealApplet.html*. This applet allows students to simulate buying cereal boxes that contain one of six possible prizes. Have your students simulate buying cereal boxes until they have received all six prizes. They could collect data from a large number of simulations and use the data to answer the question, "On average, how many boxes of cereal would a person need to buy to receive all six prizes?"

4. Use TinkerPlots or other software to collect a large number of simulations. Have the students compare their class plot to one with thousands of simulations.

References

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